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LECTURE NOTES

ON TURBULENT FLUID MOTION

by

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Hy. 200

1951

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Pasadena, California

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Chapter I

Introduction

1. It is difficult to give a clear cut definition of "turbulence." In general, we speak of turbulence when the precise form of motion of a field does not interest us and we want to know certain average or statistical aspects of this motion only. Such a situation presents itself when the motion is of a rather "irregular" character and when its general pattern repeats itself an indefinite number of times.

A very important case is that where the actual motion can be considered as fluctuating about a certain mean state so that the mean value of a quantity, taken over a certain period of time, tends to become independent of the length of the period and of the instant at which the period is made to begin. Another case presents itself when the character of the motion appears to be the same over a large part of the field. In the case of stationary fluid motion through a long tube or canal of constant section, a combination of the two aspects is found; at every point of the tube the field can be considered as stationary, while at any instant the general state of the field will be the same over every section of the tube, provided we are far enough away from the entrance and from the exit of the tube. In such cases, mean values can be taken either with respect to time at a given point, or over a straight line parallel to the axis of the tube at any instant of time. We may also combine the two methods of averaging.

In other cases we only find a stationary pattern with respect to time, but no homogeneity in space; for instance when we consider the state of motion in a pump working with constant speed of rotation,

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constant delivery, and constant pressure difference.

There can also be cases where the state of motion is the same over the whole field (so that the field is statistically homogeneous), but where the motion gradually dies out so that it is not stationary with respect to time. In that case space averages must be used .

There are, of course, very many cases where there is neither stationarity with respect to time, nor homogeneity in space. It may be, however, that the same situation can be reproduced an indefinite number of times by repeating the experiment which brought it about. In such a case we can use averages defined with respect to the series of repetitions of the experiment. Such averages are denoted as "ensemble" averages, since they refer to an ensemble of cases.

2. The number of possible types of turbulent fluid motion is, of course, infinite. Attention has been directed mainly to a few groups of types of such nature that the members of a single group appear to be related amongst each other.

The technically most important of these groups derives from the state of turbulence which is found in a long cylindrical tube with constant cross section. The statistical properties of the field are independent of the instant of time and of the coordinate x measured along the axis of the tube. In the case of a circular cross section they are moreover symmetrical about this axis, but with other forms of section the latter property falls away.

It will not be necessary to enumerate all the types of turbulence related to the one mentioned, since these are sufficiently known. In general, one can combine them under the name "boundary layer turbulence."

Since this form of turbulence is stationary in time, and since it is also known that there is dissipation of energy in consequence of viscous friction, the motion must be maintained through the introduction of energy from the outside. In the case of the motion through a tube, this energy is derived from the pressure drop in the direction of the axis of the tube. Connected with this circumstance is the fact that the statistical properties are not homogeneous throughout space: although independent of x , they are dependent on the distance of a point

from the wall, and all evidence points to the extreme importance of this dependence in the energy relations.

Turbulence, homogeneous over a large space, has never been obtained in an exact way. It is believed that it can be obtained by suitably stirring a fluid in a large space and then leaving it to itself under such conditions that the influences of the boundary of the field can be neglected in the part where we desire to investigate the turbulent motion. Theoretically this case has been extensively studied, because it has simpler properties than non-homogeneous fields, so that it is more amenable to mathematical analysis. The mathematical analysis can be extended in such a way that certain large scale inhomogeneities can be taken into consideration. It is believed that the theory gives relations which can be applied with sufficient approximation to a number of experimental cases.

These forms of homogeneous turbulence are constantly dissipating energy. The cases studied refer to fields not acted on by exterior forces (after the initial stirring); hence the turbulence will die out. The way in which it dies out forms one of the main themes of the theoretical treatment.

3. One can ask whether it would be possible to obtain fields which simultaneously are homogeneous in space and stationary in time. Such fields necessarily would need the action of exterior forces in order to entertain the motion. It will be evident, however, that the spatial pattern of these forces will influence the character of the turbulent motion. Hence homogeneity can be found only when the scale of length used in observation is so large that details of the force distribution are eliminated. Since in general one will have to apply forces which are fluctuating or act intermittently, a similar precaution will be necessary in respect to time in order that the field may be considered as stationary. It seems difficult to imagine a proper experimental set-up for such a case, but one might perhaps think of a large school of fish in the sea, distributed with constant average density and playing around in an irregular way with similar character of motion at every point. The most directly influenced aspect of the turbulence in such a case will be a type of eddies, comparable in extent with the

size and the average distances between the fish, and with periods depending on the motion of the fish and on boundary layer phenomena along their bodies. For an observer of dimensions small with respect to the fish the field will therefore not be homogeneous. But when considered over distances large compared to the mean distance of the fishes, the field will present homogeneous patterns. If we keep in mind that random distribution of the fish will present a certain amount of large scale irregularities, it will be evident that the field of turbulence, considered in the main, will present features (eddies) of dimensions large compared with the average distance between the fish. The study of these large scale irregularities may form a subject of great interest. One can, of course, replace "fish" by any system of forces in this example.

4. The type of field which has been used very extensively for investigations concerning approximately homogeneous, decaying turbulence, is the grid-produced wind-tunnel turbulence. Here we have to do with a type of turbulence produced by exterior forces, the action of which has a stationary character in time. The field obtained is likewise stationary in time, but its distribution in space is not homogeneous. To bring this case in its proper perspective with respect to the other ones, we describe it as follows: we consider the field of motion in the entrance part of a large tube; it is known that by means of proper devices the velocity distribution of the incoming current can be made homogeneous and its turbulence can be reduced to about 0.01%; while boundary turbulence, which makes its appearance at the walls, does not penetrate to the interior until one is far downstream from the entrance. In the approximately homogeneous and regular entrance current a screen or grid is introduced, with a mesh size small in comparison with the cross section of the current. This screen produces turbulence, and the eddies formed are carried along by the mean motion and gradually die out. When we introduce coordinates, x in the direction of the main motion, y and z parallel to the plane of the screen, the average state of motion at any point will be stationary with respect to time; the motion will be greatly dependent on y and z near the screen, but its average pattern will become more and more independent of y and z

further downstream; finally it will change gradually with x , but the change may be so gradual that over restricted distances we may consider the motion as being homogeneous with respect to all three coordinates. It is expected that the statistical character of the turbulence then will also become isotropic.

The type of turbulence obtained in this case is very different from that obtained in a cylindrical tube when the boundary layer turbulence has penetrated so far into the interior that a pattern of motion results which has become independent of x .

It is believed, nevertheless, that certain characteristics of homogeneous, isotropic turbulence, will be present also in the turbulence found in a tube if attention is restricted to small scale motions. One of the problems of turbulence theory is to find out in how far this is the case and in how far the differences between the two cases can be understood.

5. In the greater part of the theoretical investigations on turbulence the fluid is considered to be incompressible. The types of motion possible are, consequently, potential motion satisfying the ordinary Laplace equation and incompressible vortex flow.

Cases where changes of volume or density appear, begin to attract attention.

One case is that of a boiling liquid. In the first place, we can imagine a field in which a great number of bubbles are being formed and are disappearing again without presenting a mean motion. The appearance and disappearance of bubbles is assumed to be a random effect; the currents produced will be currents produced by sources and sinks, and by the large scale irregularities in the distribution of the growing or of the disappearing bubbles. One might find here a type of turbulence wholly of "potential" nature, though with a potential not satisfying the ordinary Laplace equation.

A second case would be that where the bubbles have a certain mean motion, and where, moreover, the frequency of appearance or disappearance is a function of the coordinate x , measured in the direction

of the mean motion. We then come to the simplest case of cavitation.

In the general case of flow with cavitation the field is three-dimensional. Already in the one-dimensional case the physical aspect of the process forces us to investigate the pressure gradient and to have regard to the effect of pressure on the appearance and disappearance of bubbles. This becomes much more difficult in the three-dimensional case.

The question also arises whether there is a loss of energy in such cases. It is possible that there would be no dissipation by viscosity so long as the motion of the liquid is purely potential and satisfies the ordinary Laplace equation outside of the bubbles. The only sources of dissipation then would be thermodynamical through some irreversibility in the processes of evaporation and dissipation, through heat conduction, or by means of sound waves, when the compressibility of the liquid is taken into account.

When compressibility of the fluid becomes of great influence, all pressure changes will set up acoustical waves so that part of the energy of the field is in the form of eddy motion, another part in the form of acoustical waves. When the field is not homogeneous and boundary conditions have to be taken into account, there may be outflow of both types of energy. Such forms of turbulence will be of importance in high velocity boundary flow and also in problems referring to stellar atmospheres or to interstellar gas.

Finally, there are forms of turbulence in which electromagnetic forces play a part and influence the dissipation.

Our list is still very incomplete. We may mention cases which can be found in the atmosphere, or in the ocean, where there are simultaneously present two forms of turbulence, one large scale, the other small scale, with a marked gap in between. The small scale turbulence will then act as a kind of eddy viscosity for the large scale motions. The nature of this action, and the comparison between this case and cases where there is a gradual passage from large scale to small scale motions, form problems of great interest.

6. There is still a further point to be considered in relation to turbulent motion. In the preceding considerations we have repeatedly spoken of turbulence produced by exterior forces. In certain cases we can imagine that these forces are given, if not as exact functions of the time and the coordinates, then at least as random functions. This was the case when we considered the fish; to a certain extent also the production of turbulence in a wind tunnel by means of a grid can be brought under this heading. We can then assume that the irregular and fluctuating character of the motions primarily is a consequence of the randomness of the force system. Since the equations of hydrodynamics are non-linear, the fluid motion cannot be treated as a simple superposition of different types, each called forward by some component of the force system; on the contrary, the various patterns of motion produced by the forces interact and give rise to new patterns which may differ widely in character, in scale, and in periods, from those immediately found in the forces. Still, in these examples, we did not consider a reaction from the field upon the force system.

It is generally assumed that the case is different with turbulence in the flow through a tube, and with boundary layer turbulence. The exterior force - in the case of motion through a tube: the pressure gradient - can produce a completely regular motion, the so-called laminar flow or Poiseuille flow. The fact that actually irregular flow is obtained is supposed to be a consequence of an inherent instability of the laminar flow, so that slight deviations from the mathematically exact pattern can lead to a complete change of the whole field. Turbulence thus can originate as it were "spontaneously."

If we look carefully into the picture, however, we may come to the conclusion that after all the two cases are not so very different. Let us describe the actual state of the field at any moment as a superposition of elementary types of motion, for instance, by means of the method of Fourier series or of the Fourier integral (although, instead of simple harmonic components, any other complete system of normalized solutions of a linear differential equation can be taken). It is always possible, at any given instant, to obtain such a resolution of the field. If the analysis is repeated at another instant, we obtain other

values for the amplitudes; we can thus describe the history of the field by means of the time dependence of the various amplitudes. The statistics of the field then reduce to the statistics of this system of amplitudes.

In cases where the behavior of a field is governed by linear equations, a form of resolution can be found in which all the amplitudes are completely independent of each other. Each of them follows its own course; it depends only on the way in which it may have been excited at an initial instant, and then either grows or decays; or it may be stimulated repeatedly but it is always independent of its companions. In the case of a non-linear system such a resolution into independent components is impossible; every method of resolution one may apply gives us a series of components whose equations of motion are interrelated and non-linear in such a way that they cannot be separated. Hence, every component is coupled with all others. If at the initial instant only one, or a few components are excited, other components will come up soon afterwards and, in general, we can expect that the whole spectrum will always arise. The laws of motion of those systems which present turbulence now seem to be such that there are always components which can rise to very great amplitudes, even if they had been very weakly stimulated. This has the consequence that very small disturbances can produce disproportionally large effects. Since it is impossible to eliminate all disturbing effects from any real case completely, we can always expect that small disturbances (e.g., slight irregularities of the incoming flow, slight disturbances of the flow outside a boundary layer) will excite some peculiarly sensitive form of motion and so give rise to the appearance of a turbulent field, with mean amplitudes of the fluctuating velocities out of proportion to the effects which called them forward.

7. The mathematical problem of turbulence in a tube (and also of related forms) is to describe the coupling between the elementary forms of motion into which the field can be resolved, and to predict the average distribution of energy between these forms. It belongs to the nature of the equations describing the development of the components, that

certain ones of them can take up energy from the pressure gradient acting in the direction of the axis of the tube. It will be evident that those components which can take up energy most easily will be the most sensitive to disturbances. If the taking up of energy from the pressure field is dependent on terms of the first degree in the amplitude, it will even appear in a linearized theory of the field, although such a theory will not bring out the coupling between the components. In certain cases it may be that the taking up of energy will appear only when attention is given to the terms of higher degree than the first. Which of the two cases presents itself in the turbulence in a tube, forms a problem which has not been fully investigated, but examples of systems can be constructed where the energy take-up is given by terms of the first degree.

It may be a general character of turbulence that types of motion peculiarly adapted to taking up energy are present in the system. When this is the case, we must expect that the average distribution of energy between the various components will be determined practically by the relations between the components themselves and will be independent of the magnitude and form of the stimulating agency, provided the stimulation is weak. We will then speak of "spontaneous" turbulence. With strong stimulation one must expect that certain types of motion directly related to the stimulating effects may become preponderant. In such a case one has a type of turbulence which cannot be considered as "spontaneous," but which passes into "stimulated" turbulence. Even in the case of turbulence produced by a grid or by the school of fish, we have to do with cases which, strictly speaking, come under the first category; for the forces excited by the rods of the grid or by the bodies of the fish are due to boundary layer effects, which again produce changes in the field of flow out of proportion to the thickness of the boundary layers. Once started, the disturbances continue to call forward new disturbances, so that the turbulence perpetuates itself. Nevertheless, for an over-all investigation of the field, we may sometimes neglect this aspect of the problem and assume random system of given forces to be the producing agency.

8. One may ask what is the use of the preceding considerations and how they are related to practical problems. The answer is that the investigation of any practical problem puts us in the position of giving information about certain statistical aspects of the field of motion. Cases immediately coming to the foreground concern the relation between pressure drop and mean velocity of flow in a tube, boundary layer friction, problems of diffusion and of heat transfer. All these depend on certain characteristics of the field, and in order to be able to give quantitative information an analysis of the field into some system of simple components is an absolute requirement. It can be surmised that the practical questions of resistance, diffusion, and heat transfer could be treated very well if we only had some knowledge about the components with a relatively large scale pattern. However, such mathematical attempts as have been made to unravel the intricacies of turbulent motion have given the impression that full information concerning the amplitudes of the coarser components cannot be obtained without giving attention to the whole spectrum. Since there is an unlimited supply of energy through the action of the pressure field, the mean energy content of the turbulent motion is not directly determined. Energy is accumulated and the accumulation goes so far that dissipation balances it. Hence, the mean energy content cannot be found without giving regard to the dissipation, and the total amount of the dissipation is very largely dependent on the small wavelength end of the spectrum. Thus, the whole spectrum must be attacked and we cannot hope to obtain satisfactory results if we should try to evade this.

The problems before the investigator divide themselves into two groups:

(a) To find the way in which practical quantities (determining, for instance, transfer of momentum of suspended material or of heat) depend on the character of the spectrum.

(b) To define the spectrum more precisely and to find the relations which govern the distribution of energy over it.

Serious difficulties present themselves in both groups of problems. Even if, for a given case - say again the flow through a

tube -- we analyze the flow into simple harmonic components and assume that we should know the mean amplitudes of all components, the magnitude of the diffusion coefficient for particles suspended in the field cannot be found. This magnitude is not fully determined by the distribution of energy over the spectrum alone. Other relations, defining the relation between the state of the system at consecutive instants, are needed. Hence an adequate description of the field of motion requires more than data about the spectrum--we must also have data on the persistence of the components.

Chapter II

Diffusion of Particles in a Turbulent Field

9. To obtain insight into the nature of those characteristics of the turbulent field, which are needed in the investigation of practical phenomena, we consider the diffusion of particles. We begin by assuming that all particles have the same density as the fluid and that they are sufficiently small in order to follow the motion of the elements of volume of the fluid without time lag. We assume that a large number of particles is followed, starting all from the same point of the fluid. If there is a mean motion in the field, the particles will be carried along by this mean motion. Let us suppose, although this will not always be the case, that the mean motion is stationary, rectilinear and uniform over a certain domain so that it simply represents a translation in a definite direction with constant velocity. By introducing a coordinate system moving with the mean flow, we can eliminate its effect so that we are only concerned with the turbulent motion superposed on it.

The turbulent motion will be in three coordinates. We restrict to only one of these coordinates, say y .

Observation of the motions of the individual particles will give data which must be reduced by statistical evaluation. We have assumed the field to be stationary and we will, therefore, be interested in the positions of the particles after a certain duration T since they have started from their origin. For any value of T , the same for all particles, we can picture the values of the coordinates y of the particles in a diagram. The distribution obtained may appear to be simply Gaussian. If this is the case, the shape of the curve can be characterized by a single parameter, for which one usually takes the average value of y^2 .

We will investigate how this mean value, which is a function of T , is related to properties of the field.

10. We consider the velocity of a single particle in the y -direction as a function of two variables; the time at which the particles started

and the duration T elapsed since that moment. Now:

$$y = \int_0^T dT' \cdot v(t_0, T') ;$$

hence, we have:

$$y^2 = \int_0^T dT' \int_0^T dT'' v(t_0, T') v(t_0, T'') ;$$

and the mean value taken over a large number of particles becomes:

$$\overline{y^2} = \int_0^T dT' \int_0^T dT'' \overline{v(t_0, T') v(t_0, T'')} .$$

We assume the turbulence to be stationary with respect to time and homogeneous in space. It will be necessary to assume the existence of exterior forces in order to realize these two conditions simultaneously, but since at the present we are concerned with kinematical relations only, this will not give rise to difficulties. The assumption has the consequence that the mean value $\overline{v(t_0, T') v(t_0, T'')}$ will depend on the time difference $\tau = T' - T''$ only, and more precisely on the absolute value $|T' - T''|$ of this difference, since the order in which the two positions are taken is immaterial. If the turbulence, although being stationary in time, would not be homogeneous, the values of T' and T'' will enter. We will keep to the simpler case where T is the only relevant variable.

We write:

$$R_v(\tau) = \overline{v(t_0, T') v(t_0, T'')}$$

One can now calculate the mean value $\overline{y^2}$ from the following integral:

$$\overline{y^2} = \int_0^T dT' \int_0^T dT'' R_v(\tau) ,$$

which can also be written:

$$\overline{y^2} = 2 \int_0^T dT' \int_0^{T'} dT'' R_v(\tau) ,$$

and can be further transformed into:

$$\begin{aligned} \overline{y^2} &= 2 \int_0^T dT' \int_0^{T'} d\tau R_v(\tau) = \\ &= 2 \left\{ T \int_0^T d\tau R_v(\tau) - \int_0^T d\tau \cdot \tau \cdot R_v(\tau) \right\} . \end{aligned}$$

The integral occurring in the first term will approach to a constant value, when T becomes large; we write:

$$2 \int_0^\infty d\tau R_v(\tau) = D .$$

Also the second integral will approach to a constant value, and we write:

$$2 \int_0^\infty d\tau \cdot \tau \cdot R_v(\tau) = D \cdot T_0 .$$

In this way we obtain:

$$\overline{y^2} = 2D(T - T_0) \quad \text{for large } T .$$

Expressions for small values of T can be obtained likewise. (They will have a more complicated form.) We will pass over the details, however.

The function, $R_v(\tau)$ is called the correlation function for the movement of the particle (more precisely, for the movement in the y -direction; other correlation functions may be needed for the movement in the x -direction or in the z -direction). This correlation function, referring to the history of a single particle, or, according to our initial supposition, to the history of a single element of volume of the fluid, is a Lagrangian correlation because it is the Lagrangian description of a field of fluid motion that attention is given to the history of the individual elements of volume.

Most of our information about the state of motion of a fluid, both that resulting from experimental investigation and that obtained from theoretical deduction, is given in the Eulerian description, where velocities are recorded as a function of coordinates fixed in space, and of time t . This description does not give attention to the history of a single element of volume.

Unfortunately, the Lagrangian correlation function R_v is not directly related to the data obtained in the Eulerian description.

11. Eulerian Correlations. Eulerian correlations can refer to relations in space or to relations in time, or to both. Let us first take the case of a homogeneous field of turbulence. We consider the product $v_1 v_2$, where v_1 refers to the point y , v_2 to the point $y + \gamma$, both for the same instant t . We calculate the mean value of the product by giving to y all values in a certain length of the y -axis, keeping γ at a fixed value. The mean result obtained will be denoted by $S_1(\gamma)$. In general, this function will depend on t , the instant for which the correlation was calculated.

If the field is stationary in time, we can calculate mean values of the type $\overline{v_1(t) v_2(t + \tau)}$, referring to a single point of the y -axis. This mean value will be denoted by $S_2(\tau)$. In general, it will depend on y .

If the field is both homogeneous in space and stationary with

respect to time, we can define a more general type of correlation by making v_1 refer to a pair of values y, t , and v_2 to a pair of values $y + \eta$, $t + \tau$. To obtain the mean value, we proceed as follows: in the y, t -plane a certain domain is chosen with its center of gravity at y_0, t_0 . The mean value is calculated as the average over the area of this domain. If the area is sufficiently large, the mean value should become independent both of the area and of the position of the center. It should, moreover, be independent of the form given to the area provided no exceptional choice is made. Usually it is supposed that the same mean value should also be obtained in the particular cases where the domain is reduced to a line of a certain length, either parallel to the t -axis (time mean) or parallel to the y -axis (space mean).

If, instead of time and space averages, "ensemble" averages are preferred, we must assume that instead of a single field $v(y, t)$ a great many similar fields are given, in each of which the value of v for a given pair of values y, t , is determined by a random process. One can then take averages over the ensemble for every pair y, t , and one will thus obtain a statistical average field. This procedure can be used also for non-homogeneous and non-stationary fields. If the field is homogeneous and stationary, the average value of any quantity obtained by ensemble averaging should be the same for every pair y, t . The ensemble mean value of the product $v_1 v_2$ with fixed values of y, t , in principle can be a function of all four of these variables. In the case of a stationary field it becomes independent of both y and t , and reduces to a function of η and τ alone, to be denoted by $S(\eta, \tau)$.

12. The introduction of the "ensemble" is helpful in making clear certain matters of principle. If the random character of v for any pair of values y, t , was expressed by giving a probability function for the values of v for that pair, say in such a way that the probability for v to exceed the value a would be:

$$P_v(a; y, t)$$

(with $P_v = 1$ for $a = -\infty$, and $P_v = 0$ for $a = +\infty$); and if this embodied all our knowledge there would be no basis to obtain a relation between simultaneous values of v at different points y . Our lack of

knowledge would be expressed by the formula:

$$S(\eta, 0) = 0,$$

for all values of η which differ from zero. In the same way, we should have

$$S(0, \tau) = 0$$

for all values of τ different from zero, since there would be no information about relations between consecutive values of v at a given point y . It will be evident that more generally we should write

$$S(\eta, \tau) = 0$$

for η or τ , or both being different from zero.

Hence, whenever the Eulerian correlation function $S(\eta, \tau)$ is not zero for some values of η or τ , this means that there exist relations between the simultaneous values of v at different points y and between the consecutive values of v at a single point. Naturally one will expect such relations to exist on physical grounds. It would be absurd to imagine that in a moving fluid the velocity could change from point to point or from instant to instant without any restriction, however close in space or in time the points or instants would be chosen. One may ask, therefore, for a description of the field, which gives a picture of such relations, and one may also expect that something about these relations must follow from the equations of motion.

In the description of the field by means of random functions, as is the method employed when an "ensemble" of fields is considered, these relations are obscured unless one introduces special probability functions referring to the simultaneous values of v at different points and to consecutive values at a single point, etc. The mathematical expression of such probability relations, however, is not simple and brings further questions. For most purposes, therefore, we consider the information embodied in the correlation function $S(\eta, \tau)$ as basic, and we shall make no attempts to derive it from probability relations.

The following problems now present themselves:

- (a) Is there a relation between the Eulerian correlation function and the Lagrangian correlation we needed for the problem of diffusion?

(b) Can one interpret the relations embodied in the function $S(\gamma, \tau)$ as the result of a certain structure present in the field? This latter problem will be deferred to Chapter III.

13. The problem concerning the relation between Lagrangian correlation and Eulerian correlation has not been solved. Not much attention seems to have been given to it in publications. It has been considered to some extent in a paper by F. N. Frenkiel, "Comparison Between some Theoretical and Experimental Results on the Decay of Turbulence," Proc. VIIth Intern. Congress of Applied Mechanics, London, 1948.

In order to pass from the Eulerian description to the Lagrangian it is necessary to integrate the differential equation:

$$\frac{dy}{dt} = v(y, t).$$

Let the integral be given by $y = \phi(t; s)$, where s is the integration constant. Every value of s will characterize an integration curve or path. Having found the integral, differentiation gives the velocity v as a function of the time t along a particular path. Since it also may be dependent on the value of the integration constant, we write:

$$v = \psi(t; s).$$

When the field is stationary with respect to time, we can now define time mean values along a particular path, that is, for a fixed value of s . In this way we come back to Lagrangian mean values. It is generally assumed that the mean value of v itself along the path, obtained in this way, is equal to the Eulerian mean value; in particular if the Eulerian mean value of v is zero, it is supposed that the Lagrangian mean value of v is zero too.*

The Lagrangian correlation is defined as the time mean value:

$$R_v(\tau) = \overline{\psi(t; s) \cdot \psi(t + \tau; s)}$$

It is evident that this function differs widely from all three Eulerian

* For an exception, see below p. 20.

correlation functions.

When the field is not only stationary with respect to time, but also homogeneous in space, the Lagrangian mean value will be the same for every path, so that it will be independent of s . One might consequently define a Lagrangian mean value also by means of an average taken with respect to both t and s . For this purpose one could imagine the function $\psi(t,s)$ to be represented in an auxiliary plane having s and t as coordinates, so that every vertical line corresponds to a particular path. We may then define mean values by averages taken over areas in this plane. In certain cases one may even attempt to obtain the mean value by averaging over a set of values of s for constant t .

In the latter concept, however, a difficulty presents itself, since the parameter s can be chosen in an infinite number of ways. This means that in the s, t -plane, we can introduce any transformation which substitutes a new variable s' for s , provided t is left unchanged. Such a transformation, however, will in general affect the definition of mean values. One might attempt to evade this difficulty by defining s as the value of y at a given instant of time, but it is not automatically certain that this will really take away the full difficulty.

We shall restrict, therefore, to time mean values taken along a single path, and will assume that these will be independent of the particular path. The introduction, afterwards, of an average with respect to s will then not bring any uncertainty.

The major problem is situated in the integration of the differential equation. The usual methods of series development do not give sufficient help, since they have a restricted domain of convergence, while the interesting aspects of the correlation function make their appearance when we consider large time differences. An important question, for instance, is: for which time difference τ will R become zero when we know the behavior of the function S ?

Questions of this nature refer to properties of the integral "in the large" according to modern terminology.

The following examples may serve as subjects for study:

$$(1) \quad \frac{dy}{dt} = A \cos(\omega t + \lambda y)$$

with $A < \omega/\lambda$;

$$(2) \quad \frac{dy}{dt} = \frac{A \cos \omega t}{1 + \cos \lambda y}$$

with $\alpha \ll 1$.

The following examples can serve as subjects for study:

$$(1) \quad v = \frac{dy}{dt} = A \cos(\omega t + \lambda y), \text{ with } A \ll \omega / \lambda ;$$

$$(2) \quad v = \frac{dy}{dt} = \frac{A \cos \omega t}{1 + \alpha \cos \lambda y}, \text{ with } \alpha \ll 1.$$

In the first example it will be found that the mean value of v along a path is not zero, notwithstanding the fact that the Eulerian mean values of v , both with respect to time and with respect to y , are equal to zero.

In the case of the second example the Lagrangian mean value of the velocity appears to be zero, the same as the Eulerian time mean value. However, the Eulerian mean value with respect to y (for constant t) is not zero.

Both examples are far too simple to represent conditions existing in turbulent fields. With more elaborate expressions for v , however, integration becomes too difficult.

14. When we investigate the form of a correlation curve two characteristics are of great importance: the point where the correlation becomes zero for the first time, and the curvature at the top. The second characteristic still belongs to the domain of "near by" relations and can be studied by means of differential expressions.

Taking the case of the Lagrangian description, we have

$$\frac{d}{dt} \left(v \frac{dv}{dt} \right) = v \frac{d^2 v}{dt^2} + \left(\frac{dv}{dt} \right)^2 ,$$

all differentiations referring to the history of an element of volume, followed during its motion. For stationary turbulence, the mean value of the left hand member will be zero, hence:

$$\overline{v \frac{d^2 v}{dt^2}} = - \overline{\left(\frac{dv}{dt} \right)^2} .$$

Now we have:

$$\frac{dR_v}{d\tau} = \overline{v(t) \left(\frac{dv}{dt} \right)_{t+\tau}} ;$$

$$\frac{d^2 R_v}{d\tau^2} = \overline{v(t) \left(\frac{d^2 v}{dt^2} \right)_{t+\tau}} ,$$

and for $\tau = 0$:

$$(I) \quad \frac{d^2 R_v}{d\tau^2} = \overline{v \left(\frac{d^2 v}{dt^2} \right)_t} = - \overline{\left(\frac{dv}{dt} \right)^2} ,$$

differentiations and mean values all referring to the Lagrangian description. This result shows that the curvature of the correlation function at its top (for $\tau = 0$) is connected with the mean value of the square of the derivative.

In a similar way for Eulerian time correlation we have:

$$(II) \quad \frac{d^2 S_2}{d\tau^2} = - \overline{\left(\frac{\partial v}{\partial t} \right)^2} ,$$

derivatives and mean values referring to the Eulerian description.

Now we have the well known relation:

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial y}$$

from which

$$\left(\frac{dv}{dt} \right)^2 = \left(\frac{\partial v}{\partial t} \right)^2 + 2v \frac{\partial v}{\partial t} \frac{\partial v}{\partial y} + v^2 \left(\frac{\partial v}{\partial y} \right)^2 .$$

However, this formula cannot be used to make a comparison between (I) and (II) because in (I) we have a Lagrangian mean value and in (II) an Eulerian one.

Chapter III

THE SPECTRUM OF A TURBULENT FIELD

15. We have seen that the Eulerian correlation function gives information about connections in the field of flow extending over space and time. This brings us to the problem of the structure of the field at a given instant. Various methods are available for analyzing the geometrical pattern of the motion. We shall consider two of them.

For simplicity, we consider a turbulent field without mean motion and assume the turbulence to be homogeneous in space. It does not make much difference if we start with a three-dimensional field, but in further work it is simpler to restrict to the one-dimensional case and to postpone three-dimensional fields to a later occasion.

One method (applied by von Weizsäcker and others) starts from the idea of taking average values of the velocities over domains of decreasing dimensions. We imagine a scale of lengths $L_0, L_1, L_2, L_3, \dots$, in which L_0 is large compared with the largest eddies which observation may disclose in the field, while every next term of the series is obtained by taking $1/2$ of its predecessor. We now associate with every point x, y, z of the field a series of velocities U_n, V_n, W_n , obtained by averaging the values of the actual velocities around this point over a volume of magnitude L_n^3 , having its center at x, y, z . It can be expected that when L_0 is large enough, the values of U_0, V_0, W_0 will be zero for every point x, y, z ; whereas more and more details of the field of motion will become apparent when we go down the scale. It must also be expected that when we have come to a certain very small length L_N , depending on the nature of the field but in all normal cases large compared with molecular distances, a further subdivision of L_N will not reveal new details. This means that there is a certain minimum scale for the turbulence, so that the motion shows coherence over distances of the order L_N or smaller, while in this process we remain still far from the separate molecular motions. (This circumstance was already brought forward by Osborne Reynolds in his classical paper on turbulence; it was considered by him to be a cornerstone of great importance in the analysis.)

All these calculations of averages refer to one single instant of time. The analysis can be performed for a series of instants of time, so that at every point of the field we may obtain U_n, V_n, W_n as functions of the time.

We now introduce a series of "component fields" defined by the formulas

$$u_0 = U_0, \quad v_0 = V_0, \quad w_0 = W_0;$$

$$u_n = U_n - U_{n-1}, \quad \text{etc.}$$

from which:

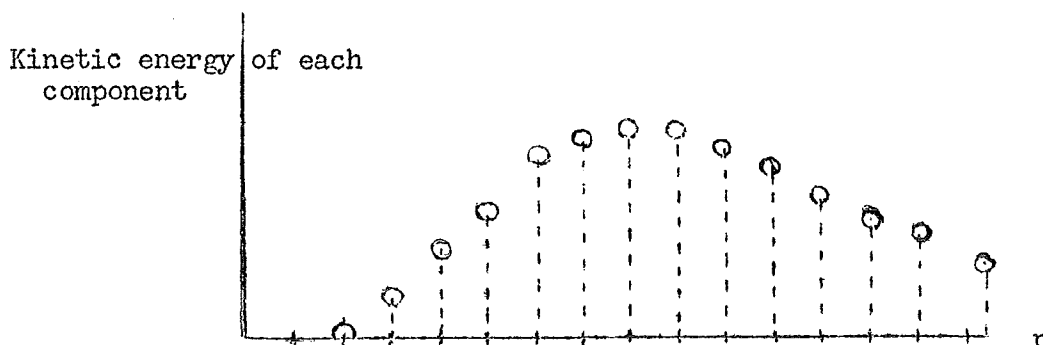
$$U_n = u_n + u_{n-1} + u_{n-2} + \dots + u_0, \quad \text{etc.}$$

The fields u_n, v_n, w_n (which quantities of course are functions of x, y, z) can be considered as "components" of the total field.

From what has been said concerning the behavior of U_n, V_n, W_n for large n (in particular for $n > N$), it follows that the component fields u_n, v_n, w_n become zero for $n > N$.

For each of the component fields we can calculate the mean kinetic energy per unit volume (assuming that the field is statistically homogeneous). This mean kinetic energy refers to a single instant of time, viz., the instant for which the mean values have been calculated. By repeating the calculation for a series of instants, the mean kinetic energy of a field u_n, v_n, w_n can be obtained as a function of the time.

By means of this procedure we obtain a picture of the distribution of the energy over the various components as a function of the index n , which gives us a kind of "energy spectrum" of the turbulent motion.



One would expect that the total energy of the actual field per unit volume could be calculated by summing the energies of all the component fields. An exact proof of this assertion is not easy, however, and it might be that the theorem is not exactly true.

16. The analysis of a given field by means of the method of averaging according to the scheme indicated, suffers from the arbitrariness involved in the choice of the domains. One can eliminate the arbitrariness of L_0 by introducing a set of increasing lengths L_{-1} , L_{-2} , ..., derived by multiplying in steps of 2. The fields obtained by averaging over the corresponding volumes should be zero, if L_0 had been properly chosen; otherwise a larger volume must be taken to start with. Instead of dividing by 2, one might also divide by some other number, either larger or smaller, perhaps by a number close to unity. In the latter case the number of component fields will greatly increase and each field will become of very small intensity. A particular mathematical method must then be introduced in order to retain definable components.

One can obviate such difficulties by substituting for the method of averaging the Fourier analysis. Restricting to a single coordinate, say y , we represent the velocity $v(y,t)$ - in the Eulerian description - by means of a Fourier integral:

$$(1) \quad v = \int_{-\infty}^{+\infty} \varphi(k) e^{iky} dk,$$

where $i = \sqrt{-1}$. Now another difficulty presents itself, since Fourier integrals can be defined only for functions which vanish at infinity in such a way that their squares admit a finite integral. To overcome this difficulty one usually assumes that we can restrict to the values of v within a finite domain, say within the part of the y -axis from $-M$ to $+M$, while outside this domain v is replaced by zero. The corresponding Fourier integral will now give the values of v within this domain only, but if M is large, the integral expression will be sufficiently representative. The amplitude function $\varphi(k)$ is then obtained from:

$$(2) \quad \varphi(k) = \frac{1}{2\pi} \int_{-M}^{+M} v(y) e^{-iky} dy.$$

The amplitude function must satisfy the relation $\varphi(-k) = \varphi^*(k)$, the asterisk denoting the complex conjugate, since otherwise v would not be a real function.

The Fourier integral refers to the course of v at a single instant of time. When the analysis is repeated at a later instant, the amplitude function φ will, in general, be different. Hence, we must consider φ as a function both of the frequency k and of the time t . Provisionally we restrict to the single instant.

17. To obtain the Eulerian correlation function $S_1(\eta) = S(\eta, 0)$ (with τ equal to zero, meaning that we consider correlation between simultaneous values only), we form:

$$v_1 v_2 = \int_{-\infty}^{+\infty} dk_1 \int_{-\infty}^{+\infty} dk_2 \varphi(k_1) \varphi(k_2) e^{i(k_1 + k_2)y + ik_2 \eta}.$$

In order to find the mean value $\overline{v_1 v_2}$ we integrate with respect to y from $-M_1$ to $+M_2$, where $M_1 > M$. Since the integrals give a value different from zero only when y is situated in the domain $-M \dots +M$, we obtain the mean value by dividing through $2M$. The choice of the wider integration limits has been introduced to be able to make M_1 go to infinity after performing the integration in order to obtain a simplified result. In this way we find:

$$\overline{v_1 v_2} = \frac{1}{M} \int_{-\infty}^{+\infty} dk_1 \int_{-\infty}^{+\infty} dk_2 \varphi(k_1) \varphi(k_2) e^{ik_2 y} \frac{\sin(k_1 + k_2) M}{k_1 + k_2},$$

which for $M_1 \rightarrow \infty$ transforms into:

$$\frac{\pi}{M} \int_{-\infty}^{+\infty} dk \varphi(k) \varphi(-k) e^{-ik y},$$

k having been written for k_1 . Hence

$$\overline{v_1 v_2} = \frac{2\pi}{M} \int_0^{\infty} \varphi(k) \varphi^*(k) \cos k y \, dk.$$

If we write:

$$(3) \quad \Gamma(k) = \frac{2\pi}{M} \varphi(k) \varphi^*(k),$$

we can now represent the correlation function by the integral:

$$(4) \quad S_1(y) = \overline{v_1 v_2} = \int_0^{\infty} \Gamma(k) \cos k y \, dk.$$

An important point to observe is that the amplitude function φ depends on M , since M occurs in the limits of the integral (2). On the other hand S_1 by its nature must be independent of M , and the same must

apply to the function $\Gamma(k)$. Hence it follows that the absolute value of the amplitude function must be proportional to $M^{1/2}$. According to (2) the value of ϕ is obtained by the addition of a large number of vectors $v(y) e^{-iky}$, in which $v(y)$ varies incessantly from positive to negative values and vice versa. The fact that the resulting vector has a length proportional to $M^{1/2}$, is related to similar results obtained in the problem of the random walk.

A further observation should be made. For convenience, the limits in (2) have been written $-M, +M$. The mean value resulting for S_1 , however, must not change if we shift the integration interval along the y -axis. Hence the absolute value of $\phi(k)$ must be insensitive to such a shift. (The phase angle of $\phi(k)$ on the other hand may be sensitive to a shift. This phase angle may also change markedly when the value of k is altered.)

If we take $\eta = 0$, we obtain the mean square of the velocity v . Passing to the mean kinetic energy and omitting for simplicity the density factor, we find:

$$(5) \quad E = \frac{1}{2} \int_0^{\infty} \Gamma(k) dk .$$

Hence the function $\Gamma(k)$ gives us the distribution of the energy over the various harmonic components, that is, what is commonly called the energy spectrum of the turbulence.

If $S_1(\eta)$ is known, we can obtain $\Gamma(k)$ from the inversion of (4):

$$(6) \quad \Gamma(k) = \frac{2}{\pi} \int_0^{\infty} S_1(\eta) \cos k \eta d\eta ,$$

so that the energy spectrum can be calculated from the correlation function.

As mentioned, all formulas of this section refer to a single instant of time. Hence the spectrum obtained, properly speaking, is an instantaneous spectrum. Repetition of the calculations for consecutive

instants of time will show whether and how the spectrum changes with time.

18. When the turbulence is stationary one expects that the spectrum will be independent of the time. This means that we expect $\Gamma(k)$ and the absolute value of the amplitude function $\phi(k)$ to have always the same value; the phase of $\phi(k)$ probably will be a rapidly changing function of the time.

In the case of turbulence which is both homogeneous in space and stationary in time, we can generalize the procedure of section 17 and represent the function $v(y,t)$ by means of a double Fourier integral:

$$(7) \quad v = \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} d\omega f(k, \omega) e^{i(ky + \omega t)}.$$

This representation shall be valid for $-M < y < +M$; $-T < t < +T$; outside of this domain it will give $v = 0$.

We can then form the complete Eulerian correlation function and obtain:

$$S(\eta, \tau) = \frac{\pi^2}{MT} \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} d\omega f(k, \omega) f(-k, -\omega) e^{-i(k\eta + \omega\tau)}.$$

This can also be written:

$$(8) \quad S(\eta, \tau) = \int_0^{\infty} dk \int_0^{\infty} d\omega [F(k, \omega) \cos(k\eta + \omega\tau) + G(k, \omega) \cos(k\eta - \omega\tau)]$$

where:

$$F(k, \omega) = \frac{2\pi^2}{MT} f(k, \omega) f(-k, -\omega);$$

$$G(k, \omega) = \frac{2\pi^2}{MT} f(k, -\omega) f(-k, \omega).$$

If we take $\tau = 0$, we come back to the function $S_1(\gamma)$; hence:

$$\Gamma(k) = \int_0^{\infty} d\omega \left\{ F(k, \omega) + G(k, \omega) \right\}.$$

If we take $\gamma = 0$, we arrive at the function $S_2(\tau)$. Writing:

$$S_2(\tau) = \int_0^{\infty} d\omega \Gamma^*(\omega) \cos \omega \tau,$$

we have:

$$\Gamma^*(\omega) = \int_0^{\infty} dk \left\{ F(k, \omega) + G(k, \omega) \right\}.$$

The function $\Gamma^*(\omega)$ gives the energy spectrum with reference to frequencies in time, whereas $\Gamma(k)$ gives the spectrum with reference to frequencies in space.

With the aid of hot-wire anemometers it is the energy spectrum referring to frequencies in time, which can be measured most easily and with high resolving power.

It must be observed that in defining the spectrum sometimes another point of view is taken. The Fourier analysis, say for the value of v as a time function at a given point of space, is made repeatedly over a set of consecutive periods of equal large duration T . Since we now work with finite periods, we can use a Fourier series instead of the integral, so that the amplitude function is replaced by an enumerable set of amplitude coefficients:

$$v(t) = \sum \left(a_n \sin \frac{2\pi n t}{T} + b_n \cos \frac{2\pi n t}{T} \right).$$

For each period T a new set of coefficients is obtained. Consequently, when we consider the coefficients a_n, b_n of the n -th component, they will show a collection of values which can be described by a probability function if the collection is large enough. In this way one arrives at

the notion of a probability function for the spectral amplitudes.

When one compares the last mentioned method with that of the Fourier integral, attention must be given to the circumstance that we now are restricting to an enumerable set of distinct amplitude coefficients, each referring to a discrete frequency; whereas in the case of the integral we use a continuous frequency scale. The mathematical relations between the two cases are not simple and require great care.

Instead of calculating Fourier series for a consecutive set of periods of duration T , we can also consider an ensemble of systems simultaneously present before us and apply to each the analysis by means of the Fourier integral. It may then be possible to compare the values of the amplitude function $\phi(k)$, for a given k , in the various systems constituting the ensemble, and we can investigate whether a probability function for the values of the amplitude function referring to a given k can be found.

From the mathematical point of view, it is of interest to observe that a double Fourier integral like (7) can also be used when the turbulence, although being homogeneous in space, is not stationary, but is damped sufficiently rapidly to make the integral

$$\int_0^{\infty} \frac{1}{v^2} dt$$

convergent ($\overline{v^2}$: space mean). We cannot form $S(\eta, \tau)$ in this case but we can form $S_1(\eta)$ as a space mean and obtain:

$$S_1(\eta) = \frac{\pi}{M} \int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} d\omega_1 \int_{-\infty}^{+\infty} d\omega_2 f(k, \omega_1) f(-k, \omega_2) e^{i(-k\eta + \omega_1 t + \omega_2 t)}$$

valid for $t > 0$.

Chapter IV

SOME EXPERIMENTAL DATA ON THE SPECTRUM AND ON CORRELATION

19. Data on the spectrum referring to frequencies in time.* Experimental investigations have mostly been made for the case of grid-produced wind tunnel turbulence, where we have to do with turbulence decaying in time, swept along by the general current in the wind tunnel (which has a constant mean velocity). For a measuring instrument fixed in space, the field is statistically stationary. While the field is actually three-dimensional, the measuring instrument practically follows one component only in its dependence on the time at a fixed point, say $u(t)$. The corresponding frequency spectrum $F(n)$ is defined as the fraction of u^2 contained in the (time) frequency band $n, n + dn$, and is often called the "power spectrum". The corresponding Eulerian time-correlation function, in our notation: $S_2(\tau)$, is then given by:

$$S_2(\tau) = \overline{u^2} \int_0^{\infty} dn F(n) \cos 2\pi n \tau .$$

Sometimes a function $\Psi(\tau)$ is used, defined by:

$$\Psi(\tau) = \frac{S_2(\tau)}{\overline{u^2}} = \int_0^{\infty} dn F(n) \cos 2\pi n \tau .$$

Dryden already found that the measurements of $F(n)$ for a certain range of frequencies can approximately be represented by the function (H.L.Dryden, "A Review of the Statistical Theory of Turbulence," Quart. Appl. Mathem. 1, p. 35, 1943):

$$(a) \quad F(n) \approx \frac{A}{1 + c^2 n^2} .$$

*In preparing this section use has been made of a report on the Spectrum of Isotropic Turbulence by H.W. Liepmann, J. Laufer and Kate Liepmann.

If this is used up to $n \rightarrow \infty$ the corresponding correlation function becomes:

$$(b) \quad \Psi(\tau) = e^{-2\pi\tau/c}.$$

Formula (a), however, cannot be valid for large values of n . Nevertheless, Liepmann's measurements showed that by far the largest part of the turbulent energy is contained in a portion of the spectrum for which the simple expression gives a good approximation.

It is probable that the formula neither will be valid down to $n = 0$, since finite values of $F(n)$ down to $n = 0$ would seem to imply the presence of correlations extending to indefinitely increasing time intervals τ . Experimentally, most measuring appliances cannot go far below frequencies of 1 per second, so that no reliable data are available for periods of a minute, or say an hour or more.

20. Other experiments have shown that the probability distribution of $u(t)$ - referring to its instantaneous values without giving any attention to correlation ^{is} Gaussian, so that the probability of $u(t)$ having a value between u and $u + du$ is given by:

$$(c) \quad \frac{1}{\sqrt{2\pi u^2}} e^{-u^2/2u^2} du.$$

It is proved in papers on the theory of the Brownian movement that a process which gives a Gaussian distribution for instantaneous values, and which has a correlation function of the simple exponential type (b), mentioned above, must be a "Markoff" random function. Such a function is characterized by the following property: suppose that we know the value of u at an instant t_0 ; let it be u_0 . The probability distribution of the values at a later instant t is then completely determined, so that it can be described by a function $P(u, t; u_0, t_0)$. Data concerning the value of u at instants before t_0 are irrelevant. The function $P(u, t; u_0, t_0)$ therefore is the full expression of all that we can say about correlation in the behavior of $u(t)$.

The function $P(u, t; u_0, t_0)$ must satisfy an important condition. If the value of u at t_0 is given, and we desire to know the probability

Distribution for the values at t , we can choose an intermediate instant t_1 and first write down the probability distribution for that instant:

$$P(u_1, t_1; u_0, t_0) .$$

The probability distribution for the instant t will then follow from the integral

$$\int_{-\infty}^{+\infty} P(u, t; u_1, t_1) P(u_1, t_1; u_0, t_0) du_1 .$$

The value of this integral must be identical with $P(u, t; u_0, t_0)$.

It will be evident that the distance between the two consecutive instants ($t_0 - t_0$; or $t - t_1$ and $t_1 - t_0$) in this reasoning may not be taken too small. Otherwise one could expect that the value at the instant t might be found with reasonable accuracy from the value at t_1 together with the rate of change at t_1 , which actually would mean that we have made use of data concerning two preceding instants. If, however, $t - t_1$ exceeds a certain threshold, data concerning the rate of change of u at the instant t_1 can be irrelevant for the prediction of future values, which means that the course of $v(t)$ can change in an unknown way between t_1 and t .

It is important to observe that a correlation function of the simple exponential type (b) is characteristic for processes in which the variable quantity changes abruptly with a finite amount at random instants of time. The mean interval between consecutive changes is related to the parameter c in the exponent. The abruptness of the change is evident from the fact that $\Psi(\tau)$ has a non-zero derivative for $\tau = 0$. Actually this is impossible in any real physical process. A correlation function for a physical quantity must always present a rounded top (with horizontal tangent and non-zero radius of curvature) at $\tau = 0$ and the spectral function $F(n)$ must decrease much more rapidly for large n than is indicated by the approximation (a).

With reference to turbulence, this means that an important part of the phenomenon proceeds as if elements of volume with different values of the velocity n follow each other in an irregular way, with very thin

transition regions separating them. When a transition region sweeps over the measuring instrument, the rate of change of u is very high; in between the transition regions the rate of change is low. Closer inspection will reveal that the transition regions have a certain (very small) finite thickness; $\partial u / \partial t$ does not exceed a certain order of magnitude and $F(n)$ probably will decrease as e^{-Cn} for $n \rightarrow \infty$, C being some constant.

21. It will not be strange that, although the probability function for $u(t)$ is Gaussian, as indicated in (c), the corresponding function for $\partial u / \partial t$ is not Gaussian. A measure for the deviation is given by the dimensionless quantity:

$$T = \frac{\overline{(\partial u / \partial t)^4}}{\left[\overline{(\partial u / \partial t)^2} \right]^2}$$

For a Gaussian distribution one would have $T = 3$. Actual values are larger: A.A. Townsend, "Correlation Derivatives in Isotropic Turbulence," Proc. Cambridge Philos. Soc. 43, p. 560, 1947, gives values ranging from 3,32 to 3,49, and Batchelor in later work (G.K. Batchelor, "Recent Developments in Turbulence Research," Proc. VIIth Congress of Appl. Mechanics, London, 1948) assumes the value 3,5. (For u itself, Townsend mentioned values ranging from 2,92 to 2,99, fairly in agreement with the theoretical value 3).

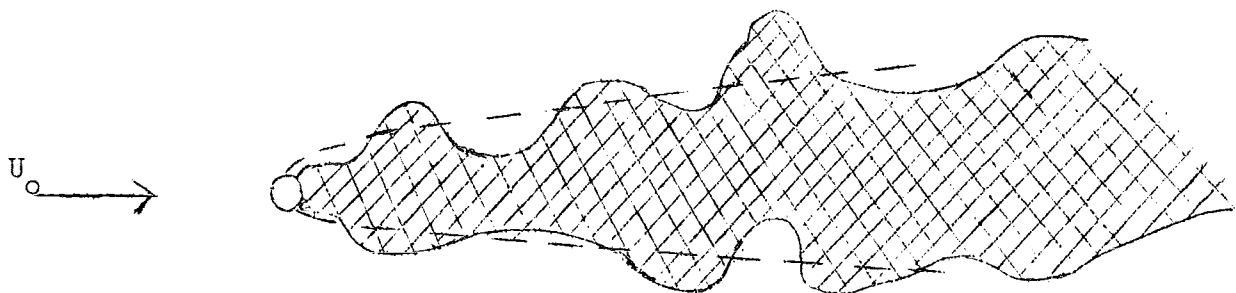
There even is no symmetry between positive and negative values of $\partial u / \partial t$. Although the mean value of this quantity is zero, the mean value of the third power $(\partial u / \partial t)^3$ in general appears to differ from zero. A dimensionless measure of the deviation is the skewness factor:

$$S = \frac{\overline{(\partial u / \partial t)^3}}{\left[\overline{(\partial u / \partial t)^2} \right]^{3/2}}$$

for which the value - 0,39 has been obtained with isotropic turbulence (Batchelor, l.c.).

Both the almost abrupt changes in u and the skewness presented by its derivative are due to the non-linear character of the hydrodynamic equations, to which we shall come back later. (The non-linear effects primarily produce a skewness in $\partial u / \partial x$; in the experimental setup this is measured as $(1/U_0)(\partial u / \partial t)$).

22. As a particular form of turbulence, we mention the turbulence in the two-dimensional wake behind a single cylinder. This may be considered as the elementary form compared to which the grid-produced turbulence in a wind tunnel is a combination and superposition of a great many wakes with different planes of symmetry. In the wake behind a single cylinder there appears a strong intermittent type of velocity fluctuation. For a certain fraction of the time the velocity varies in the ordinary irregular manner, as is characteristic for well developed turbulence, while for the remainder of the time the velocity fluctuations are slow and of small magnitude (G.K. Batchelor, "Note on Turbulent Free Flows," Journ. Aeron. Sciences 17, p. 441, 1950). It looks as if there is a rather sharp division of the flow field into a laminar flow outside a wholly turbulent wake core with an irregular boundary, while within the core the turbulence has small-scale isotropy. At any point of the wake laminar and turbulent flow will occur intermittently as the irregularities of the core are carried downstream.



23. The experimental results considered thus far have been obtained with fixed measuring instruments, and consequently refer to the Eulerian description. In deriving certain formulas connected with the results, it even has been assumed by the authors that the Lagrangian derivative du/dt would be small compared with the local Eulerian derivative $\partial u / \partial t$, and that one might approximately write $\partial u / \partial t \cong -U_0 (\partial u / \partial x)$, U_0 being the constant velocity of the general current in the wind tunnel. Hence the

results deduced from these measurements do not give light on Lagrangian correlation; the assumption made entails only that it shall stretch over much larger intervals of time than the local Eulerian correlation $S_2(\tau)$.

Measurements referring to the Lagrangian point of view have been made by following the motion of individual particles carried along by the fluid. When the particles have a density equal to that of the liquid and are small compared with the size of the eddies which form the main part of the turbulence, it can be presumed that they will closely reproduce the motion of the elements of volume of the fluid. For particles starting from a fixed point in a flowing stream (either a wind tunnel current or a current in a canal with boundary layer turbulence), measurements have been made concerning their lateral displacements after they have moved downstream over a certain distance. These measurements have given the result that the transverse displacements are distributed according to a Gaussian curve. Further, the increase of the mean square transverse displacement with distance downstream, which approximately means increase with time elapsed since a particle was ejected from its orifice, permits to calculate the diffusion coefficient D considered in section 10. This gives the time-integral of the Lagrangian correlation function $R_v(\tau)$. More detailed measurements for small distances (small time lapse) permit to calculate the correlation function $R_v(\tau)$ itself. The result obtained could be approximately represented by

$$R = e^{-\tau U_0/a},$$

where a is a constant (compare A. A. Kalinske and C. L. Pien, "Eddy Diffusion" Industrial and Engineering Chemistry 36, p. 220, 1944).

This result seems to indicate that the motion of the elements of volume of the fluid is likewise subject to almost abrupt random changes in velocity.

24. No data are known thus far, giving coexisting curves for the Eulerian and the Lagrangian correlation functions, referring to the same case of turbulent motion. It is very desirable that such measurements shall be made.

In the present case, where we consider particles carried along by a stream of approximately constant velocity, the mathematical relations

are different and somewhat simpler than those considered in sections 13 and 14. If the general motion is in the x-direction and again has the velocity U_0 , we may consider the transverse velocity v and the transverse displacement y of a particle as functions of the time t and of the downstream position x of the particle. We can then consider Eulerian correlation between particles observed at the same instant of time and finding themselves at various values of x , and thus determine a correlation function for the transverse velocities

$$S_1(\xi) = \frac{\overline{v(x,t) v(x+\xi, t)}}{\overline{v^2}},$$

where ξ is the difference of the x-values. (We now divide all correlation functions by the mean square velocity $\overline{v^2}$, in order to have normalized expressions which take the value unity for $\xi = \tau = 0$.) Or we may consider all particles passing through a fixed plane ($x = \text{constant}$), and determine a correlation function

$$S_2(\tau) = \frac{\overline{v(x,t) v(x, t + \tau)}}{\overline{v^2}},$$

depending on the time-difference. Finally, we can determine a general Eulerian correlation function

$$S(\xi, \tau) = \frac{\overline{v(x,t) v(x+\xi, t + \tau)}}{\overline{v^2}},$$

by comparing particles having time and position differences. This function can be represented by lines of constant S in a diagram having ξ and τ as coordinates.

If we follow the history of every single particle, then also a Lagrangian correlation function can be determined referring to the life history of a particle, or, what comes to the same, to the life history of a single element of volume of the fluid. We shall write for this function

$$R_v(\tau) = \frac{\overline{v(t) v(t + \tau)}}{\overline{v^2}},$$

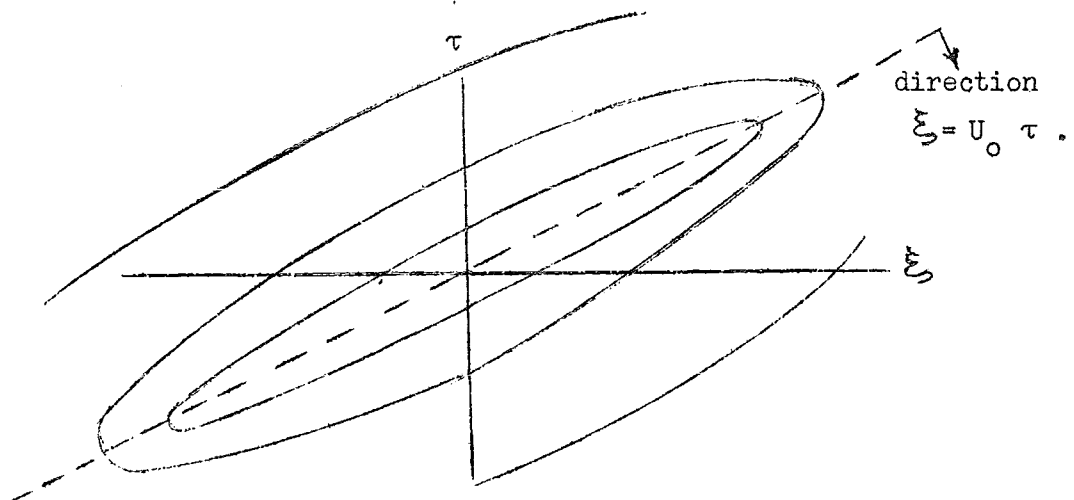
the two values of v referring to the same particle.

In the case considered, the Lagrangian time derivative of the transverse velocity is approximately given by

$$\frac{dv}{dt} \approx \frac{\partial v}{\partial t} + U_0 \frac{\partial v}{\partial x}.$$

To the same approximation the Lagrangian correlation function $R_v(\tau)$ will approach to the Eulerian correlation function $S(\xi, \tau)$, if we take $\xi = U_0 \tau$ in the latter.

If the Lagrangian time-derivative is small in comparison with the local Eulerian time-derivative $\partial v / \partial t$, the Lagrangian correlation function $R_v(\tau)$ will decrease from its maximum value unity at a much slower rate than the Eulerian time correlation function $S_2(\tau)$. We may then expect that the diagram for the function $S(\xi, \tau)$ will be somewhat of the shape:



25. Some General Observations on Experimental Data

By means of the hot-wire technique, Eulerian mean values and correlations can be obtained as time mean values in a stationary field, with measuring apparatus held in a fixed position. We mention that, for instance, the following quantities can be measured: the behavior of the longitudinal velocity component u and of the transverse components v and w as functions of time; the mean squares of these components and their spectra; correlation

products referring either to a single component or to two different components, measured simultaneously at different points of the field; the correlation product for a single component in which the two values refer to different instants of time, measured at the same spot. (This requires recording the component and reading the record by means of two instruments which pick up the values for two instants with a prescribed interval of time between them.)

It is also possible to measure directly triple correlation products, in which the square of one component is multiplied by the first power of a component measured at a different spot.

Further, one can obtain the local time derivatives of the components and measure their mean square values and analyze their spectra. Also mean square values of higher derivatives can be obtained.

Lagrangian correlations can be obtained only by the observation of particles carried along by the fluid.

The two types of turbulence for which most measurements have been made are grid-produced turbulence in a wind tunnel or the like, and the turbulent motion in a tube or canal, or in a boundary layer.

In the first case the field is stationary; over a large space its average structure is independent of the transverse coordinates y and z ; it is slowly changing with x .

In the case of motion through a tube or channel with cylindrical or prismatic walls, the field will be stationary and independent of x . There is, however, a strong dependence on the transverse coordinate which is connected with a transmission of energy across the field to compensate the dissipation. In the experimental investigation of the field one must give due attention to the fact that it is not homogeneous. This will be felt in particular when observations are made on the motion of particles, in order to obtain Lagrangian correlations, since the motion transverse to the general direction of the stream will bring them into regions where the state of the field is different.

It should not be forgotten that there are many more cases of turbulence apart from the two mentioned, several of which likewise are of technical

importance; for instance, the problem of mixing. Suppose that in a closed space a fluid is at rest; a certain motion is introduced by stirring, etc. How shall one obtain a complete mixing of the fluid in such a way that admixtures introduced locally will become distributed evenly over the whole mass? One may consider either continued stirring, or an initial period of stirring, after which the field is left to itself. In such a case the motions do not directly interest us, except in so far as it costs energy to produce them; what is of practical interest is the distribution of admixed material. In the case of an electrolyte added to the water, measurement of the electric conductivity as a function of the time at a number of spots can give a picture of the degree of homogeneity obtained and of the time needed for obtaining it.

The mathematical problem appearing here has some resemblance to the so-called ergodic problem. Can one expect that a given element of the fluid will follow a path which will practically bring it to every region of the field? To make the problem more precise, we consider the motion of a particle of the same density as the fluid; further, we divide up the field into cells of equal volume ω and ask whether the particle will ultimately pass through all these cells or whether there will be a preference for certain groups of cells, whereas others perhaps might not be reached, only very infrequently. The volume given to the cells will influence the result; (technical points of view may sometimes determine a convenient size, not too small).

When, instead of the particle, we consider some dissolving diffusing substance, ordinary diffusion or perhaps small-scale convection may ultimately bring about homogeneity.

This type of problem brings us still further from the Eulerian correlations than the Lagrangian correlations. The more so since the quasi-ergodic problem presents itself only with fields which, in the main, are stationary, but also with fields which gradually damp out. Since large scale eddies may have long times, they could promote mixing efficiently perhaps long after the motion had been initiated.

26. Physical Interpretation of the Relation between the Spectrum and the Correlation Function. In deducing Eq. (3), Section 17, for the spectral function $\Gamma(k)$ giving the distribution of the energy of the harmonic components of the field, we made use of a mathematical artifice in order to go around certain difficulties connected with Fourier integrals. It may be useful to consider the experimental determination of the spectrum in order to see why a similar difficulty is not encountered there. We take the case of a function of the time, since this is the more common problem, which can be handled much more easily than would be the case with the spatial spectrum of a function of the coordinates. The simplest example refers to the velocity $u(t)$ as measured with the aid of a hot-wire anemometer at a fixed point of the field.

The electrical signal obtained from the hot-wire anemometer, after having been amplified, is passed through an electric filter, adjusted so as to transmit only frequencies within a band of limited width. The transmitted signal can be applied to a thermo-cross, by means of which its mean square can be found. Now an electric filter is a combination of inductances, capacities, and resistances. With the incoming electric signal $u(t)$, we shall write $w(t)$ for the outgoing signal. The relation between them can be calculated from the circuit and is usually expressed by means of a differential equation; for instance:

$$\frac{d^2 w}{dt^2} + 2p\omega \frac{dw}{dt} + \omega^2 w = \alpha \frac{du}{dt},$$

where ω , p and α are quantities depending on the circuit. It has been supposed that the incoming signal operates through induction, so that what really comes in is its time derivative.

In the particular case where the incoming signal would be a pure harmonic wave:

$$u(t) = A e^{int}$$

we find:

$$w = \frac{ianA e^{int}}{(\omega^2 - n^2) + 2ipn\omega}.$$

Hence, if p is small (actually we take p to be small compared with unity), the only frequencies which are transmitted through the filter are those which differ only slightly from ω . We can say that ω determines the center of the band of frequencies which can pass through the filter, while the band width appears to be proportional to $p\omega$. If we take $n = \omega$, the expression reduces to:

$$w = \frac{\alpha}{2p\omega} A e^{\int \dots}$$

Hence in order to have a constant scale factor, one must make α proportional to ω .

When an arbitrary time function $u(t)$ is used as incoming signal, the outgoing signal is given by the integral:

$$w = \frac{\alpha}{\sqrt{1-p^2}} \int_0^\infty dt_1 u(t-t_1) e^{-p\omega t_1} \cos(\omega t_1 \sqrt{1-p^2} + \epsilon),$$

where ϵ is defined by $\sin \epsilon = p$. We use this integral to calculate the mean value of $w(t) w(t+\tau)$, which is a correlation function for the outgoing signal. The result is given by:

$$\overline{w(t) w(t+\tau)} = \frac{\alpha^2}{1-p^2} \int_0^\infty dt_1 \int_0^\infty dt_2 \overline{u(t-t_1) u(t+\tau-t_2)} \cdot e^{-p\omega(t_1+t_2)} \cos(\omega t_1 \sqrt{1-p^2} + \epsilon) \cos(\omega t_2 \sqrt{1-p^2} + \epsilon)$$

which can be transformed by introducing $\zeta = t_2 + t_1$ and $\delta = t_2 - t_1$ as new variables. The following expression is obtained:

$$\overline{w(t) w(t+\tau)} = \frac{\alpha^2}{4p\omega\sqrt{1-p^2}} \int_0^\infty d\delta \left\{ S_2(\tau+\delta) + S_2(\tau-\delta) \right\} e^{-p\omega\delta} \cos(\omega\delta\sqrt{1-p^2} + \epsilon),$$

where $S_2(\tau) = \overline{u(t) u(t+\tau)}$, that is the Eulerian time correlation for

$u(t)$ at a fixed point of space, as considered before in section 11. In this way the correlation function for the outgoing signal w has been expressed by means of a correlation function for the incoming signal.

27. We can make the following use of this result. We first take $\tau = 0$, so that on the left hand side we obtain the mean square value of w , which can be measured directly with the aid of a thermo-cross. If p is so small that $\cos \epsilon = \sqrt{1-p^2} \cong 1$, we find:

$$\overline{w^2} = \frac{\alpha^2}{2p\omega} \int_0^{\infty} d\delta \quad S_2(\delta) e^{-p\omega\delta} \cos(\omega\delta + \epsilon).$$

The integral appearing here is related to integrals considered before. In section 18 we wrote:

$$S_2(\tau) = \int_0^{\infty} d\omega \quad \Gamma^*(\omega) \cos \omega \tau,$$

the inversion of which is:

$$\Gamma^*(\omega) = \frac{2}{\pi} \int_0^{\infty} d\tau \quad S_2(\tau) \cos \omega \tau.$$

It was stated that $\Gamma^*(\omega)$ gives the energy spectrum with reference to frequencies in time. We now write:

$$\Gamma_I(\omega) = \int_0^{\infty} d\tau \quad S_2(\tau) e^{-p\omega\tau} \cos \omega \tau;$$

$$\Gamma_{II}(\omega) = \int_0^{\infty} d\tau \quad S_2(\tau) e^{-p\omega\tau} \sin \omega \tau.$$

For $p \ll 1$, Γ_I will differ only slightly from Γ^* , so long as ω is not so large that $p\omega\tau$ will become comparable to 1 in the range where $S_2(\tau)$ has not yet dropped to zero. Usually one may expect that Γ_{II} will be smaller than Γ_I . If one now arranges the circuit so that α is proportional to $\omega\sqrt{p}$, we find:

$$\overline{w^2} \cong (\text{const. factor}) \cdot \left\{ \Gamma_I(\omega) - p \Gamma_{II}(\omega) \right\}$$

In this way we see that, provided the filter is sufficiently selective and its scale factor is appropriately regulated, the mean square amplitude of the outgoing signal is nearly proportional to the spectral intensity $\Gamma^*(\omega)$.

The signal transmitted by the filter is approximately a harmonic function of the time with frequency ω . This is seen if we consider the correlation $\overline{w(t) w(t+\tau)}$ for a large value of τ , conveniently chosen so that $S_2(\tau+\delta) \cong 0$ for $\delta > 0$. We then obtain:

$$\overline{w(t) w(t+\tau)} \cong \frac{\alpha^2}{2p\omega} e^{-p\omega\tau} \left\{ \Gamma_{III}(\omega) \cos(\omega\tau + \epsilon) - \Gamma_{IV}(\omega) \sin(\omega\tau + \epsilon) \right\}$$

where:

$$\Gamma_{III}(\omega) = \frac{1}{2} \int_{-\infty}^{+\infty} d\tau S_2(\tau) e^{-p\omega\tau} \cos \omega\tau;$$

$$\Gamma_{IV}(\omega) = \frac{1}{2} \int_{-\infty}^{+\infty} d\tau S_2(\tau) e^{-p\omega\tau} \sin \omega\tau.$$

For sufficiently small p we may usually expect that $\Gamma_{III} \cong \Gamma^*$, while Γ_{IV} will be small of order p .

We thus see that w shows a nearly periodic correlation, with frequency ω , gradually damped out through the factor $e^{-p\omega\tau}$.

Since we did not make any supposition about the character of the

signal $u(t)$, apart from assuming that its correlation function $S_2(\tau)$ will become zero for τ exceeding a certain limit (this is necessary in order to make the integrals for Γ_{III} and Γ_{IV} convergent), we may say that the filter produced an almost periodic signal $w(t)$, with frequency ω determined by the properties of the circuit.

These considerations may give another proof of the fundamental nature of the correlation function. They show, moreover, that the problem whether the incoming signal $u(t)$ should be considered as a superposition of an infinite number of harmonic components or as an arbitrary irregular function, is rather irrelevant.

PROBLEMS OF TURBULENT SPREADING OF PARTICLES,
OF HEAT, AND OF MOMENTUM

28. Spreading of Particles. When "particles" were considered in the preceding chapters, we assumed them to have the same density as the fluid and to be so small that their motion would give a satisfactory picture of the motion of elements of volume of the fluid. Spreading due to random motions could then be described by introducing the Lagrangian correlation for the velocity of these elements.

We now pass to particles with a density different from that of the fluid. Two phenomena present themselves. In the case of a field extending in the vertical direction, the particles will obtain a proper motion through gravity, either falling when their density exceeds that of the fluid, or rising in the opposite case. Second, the particles will not exactly follow the variations of the motion of the elements of volume in which they find themselves; a relative velocity appears and it is necessary to give attention to the forces which the particles experience from the surrounding fluid, and conversely, to the reaction of the particles on the fluid.

The force experienced by a particle depends: on the difference in velocity between the particle and the surrounding fluid, on its size, its shape, its position with respect to the vector of the relative velocity, and on the density and the viscosity of the fluid; moreover, when the motion is variable in time, the accelerations (both of the fluid and of the particle) enter into the formulas, and an exact description may require data referring to the previous history of the motion.

It is only in the case of a spherical particle and for Reynolds numbers, small in comparison with unity, that an equation has been given expressing the resistance in function of these variables (Basset, Boussinesq, Oseen; we refer to C. M. Tchen, Mean value and correlation problems connected with the motion of small particles suspended in a turbulent fluid, Thesis Delft, 1947, Chapter 4). This equation can be integrated if the motion of the fluid surrounding the particle is known as a function of time.

Tchen arrived at the following results: (1) if the Reynolds number is small compared with unity, a constant velocity (either of fall or of rise) due to gravity acting on the difference in density, can be separated from the rest of the motion; (2) if we eliminate the motion produced by gravity, by deducting its constant velocity from the actual velocity of the particle so that there remains only the random part of this velocity, it is found that the integral of the correlation function for the random velocity component $w(t)$ in a given direction

$$D = \int_0^{\infty} d\tau \frac{w(t) w(t + \tau)}{w(t)^2}$$

is equal to the integral of the correlation function for the velocity component (in the same direction) of the random motion of the fluid surrounding the particle. Since the integral of the correlation function, according to the equations of section 10, determines the rate of spreading of a cloud of particles, this result links the spreading of the particles to correlations existing in the motion of the fluid.

The correlation function for the motion of the fluid surrounding the particle, however, is not the same as the Lagrangian correlation for the motion of an element of volume of the fluid, since, as has been observed, a particle does not exactly follow the motion of an element of volume. In general the particle will lag behind the motion of the element and its mean square velocity will be smaller than that of the element. If the element is of large size, compared with the wanderings of the particle relative to the center of the element, the particle on the whole may remain within the element; in that case the correlation for the fluid surrounding the particle will be practically identical with the Lagrangian correlation for the element. However, if the particle often passes out of the element and penetrates into a neighboring one having a different velocity, we must expect that the correlation for the motion of the fluid surrounding the particle will be smaller than the Lagrangian correlation for a single element.

29. There exists at present no method for calculating exactly the effect of the passage of a particle from one element of volume into

another one on the correlation function for the particle. The picture which we have used, of more or less well defined elements of volume of the fluid, each having a movement of its own, is already an approximation. Although it helps to visualize what is happening, we must not forget that elements of volume change form and that at their boundaries there are transition regions between the motions of adjacent elements. Hence superposed on what we have considered as "the motion of the element" there are disturbing small scale motions; and if a particle comes near to a transition region, it may be caught by these small scale motions. There are, consequently, a number of effects difficult to define, which can bring about the passage of a particle from a given element of volume into a neighboring one. In view of their random character, to a certain extent these effects can be described as a process of diffusion. In the case of very small particles, diffusion produced by molecular motions (as described by the classical physical concept of diffusion) must likewise be taken into account.

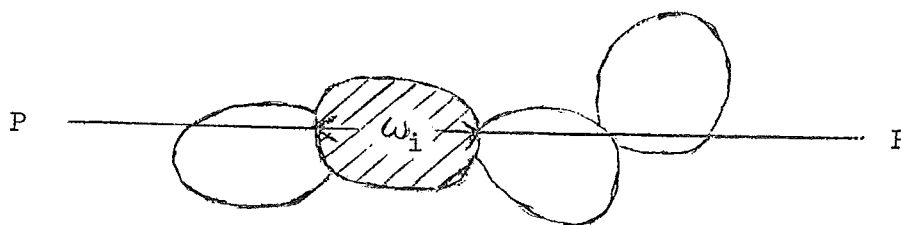
In view of this situation a different method of treatment has been developed which is applicable when we consider not the motions of a few selected particles, but the behavior of a large number of particles and wish to investigate the effect of random turbulent movements on their spreading.

We again make use of the picture of more or less individualized elements of volume of the fluid which are constantly being shuffled about in consequence of the turbulence. We suppose that the fluid contains a large number of particles of identical nature, and denote by \underline{n} the number of particles per unit of volume of the fluid (actually: volume of the medium consisting of the fluid plus the particles embedded in it). This number may be different for the various elements of volume, and for every element it can be a function of the time.

If the particles are all similar and are sufficiently small in order that we may use linear equations for their motion, they will obtain a constant velocity of fall (or rise) under the influence of gravity, to be denoted by V_g . According to Tchen's result quoted before, this constant motion can be separated from the random motions. If the

particles are of different sizes and shapes, the various types may have different velocities of fall and must be treated separately; we will keep, however, to the supposition of uniformity.

We consider a horizontal plane PP, at a given level z , in the field. At a certain instant t , this plane will cut through a number of elements of volume. Let us indicate the areas of the intersections by ω_i . If we know the particle density n_i , for each element of volume, the mean particle density over the horizontal plane PP



will be given by the sum $\sum \omega_i n_i$, taken over unit area of PP. This can be written as \bar{n} , where the bar over n denotes the mean value over the plane PP. Since gravity gives a vertical velocity of fall V_s to every particle, there will be a current of particles downward through the plane PP, of intensity $V_s \bar{n}$ per unit area and in unit time.

Superposed on this regular downward motion is the effect of the random motions. If the elements at the instant considered have vertical velocities w_i (positive if directed upward), there will be an aggregate upward transport of particles amounting to

$$\sum \omega_i n_i w_i = \overline{n w}$$

per unit area and in unit time. We must find out how this transport through random motions can be related to quantities characterizing the average distribution of the particles over the field and the intensity of the turbulence.

30. For this purpose it is necessary to give attention to the exchange of particles between adjacent elements of volume. This exchange will influence the value of \bar{n} for an element, if the average particle density in the surrounding elements is different. Let us denote this

average "surrounding particle density" by \underline{n}^* . The rate of change of \underline{n} due to exchange of particles, with a certain degree of approximation, can be put proportional to $\underline{n}^* - \underline{n}$, as follows:

$$\frac{d \underline{n}}{d t} = \lambda_1 (\underline{n}^* - \underline{n}).$$

Here λ_1 is a coefficient depending on a number of unknown factors, amongst which play a part the dimensions of the element considered and the intensity of the small scale motions (eventually including classical diffusion due to molecular effects). The factor λ_1 may be different for elements of different size, but if we classify the elements we may suppose that λ_1 will have approximately the same value for elements of equal size. A general average can then be taken at a later stage of the calculations.

We had already introduced the average value \bar{n} of the particle density over the plane PP, which the element we are considering is just crossing. This general average value need not be the same as \underline{n}^* . The two quantities, however, cannot differ much. The average over the plane PP includes a contribution from the element under consideration (with density \underline{n}), a large contribution from its surroundings (with density \underline{n}^*) and a contribution from further elements. It seems possible under these circumstances to write:

$$\bar{n} = a \underline{n}^* + (1 - a) \underline{n},$$

where a is a coefficient probably not differing much from unity. We can then transform the expression for dn/dt into:

$$(1) \quad \frac{d \underline{n}}{d t} = \lambda (\bar{n} - \underline{n}),$$

where $\lambda = \lambda_1/a$ is a new coefficient, which henceforth will be used instead of λ_1 . It can be roughly assumed that λ is inversely proportional to the square of the diameter of the element.

Equation (1) can be used to find how the particle density \underline{n} in the element under consideration has appeared as the result of exchanges with neighboring elements during its past history. The present value of

\underline{n} can be expressed by the integral of (1):

$$(2) \quad n = \lambda \int_0^{\infty} dt' e^{-\lambda t'} \bar{n}(t - t')$$

Here $\bar{n}(t - t')$ indicates the mean value of the particle density at the level where the element found itself at the instant $t - t'$. To find this particle density, we assume that the average particle density is stationary (independent of the time), and that, in the neighborhood of the plane PP, it can be considered as a linear function of \underline{z} , say,

$$\bar{n} = C + bz.$$

The level \underline{z} where the center of the element finds itself at the instant $t - t'$ can be found by means of an integration of the vertical velocity \underline{w} of the element:

$$z = z_P - \int_0^{t'} w(t - t'') dt''$$

It follows that the value of \bar{n} for the level at which the element found itself at the instant $t - t'$, is given by:

$$(3) \quad \bar{n}(t - t') = (C + bz_P) - b \int_0^{t'} w(t - t'') dt''.$$

31. When the expression (3) is substituted into (2) we obtain:

$$\begin{aligned} n &= (C + bz_P) - b \lambda \int_0^{\infty} dt' e^{-\lambda t'} \int_0^{t'} w(t - t'') dt'' \\ &= (C + bz_P) - b \int_0^{\infty} dt' e^{-\lambda t'} w(t - t'). \end{aligned}$$

The first term on the right hand side is nothing else than the average particle density \bar{n} at the level of PP where the element finds itself at the instant \underline{t} ; it is independent of the particular element of volume

under consideration. We may therefore write:

$$(4) \quad n - \bar{n} = - \frac{d\bar{n}}{dz} \int_0^{\infty} dt' e^{-\lambda t'} w(t - t'),$$

where b has been replaced by $d\bar{n}/dz$.

We must introduce this expression into the formula \overline{nw} for the transport due to random motions. Since in a field of homogeneous turbulence the assumption of random motions entails that there is no average flow, the mean value of w itself over the plane PP must be zero (incompressibility being presupposed). Hence, in determining \overline{nw} , the term \bar{n} in (4) drops out and there remains:

$$(5) \quad \overline{nw} = - \frac{d\bar{n}}{dz} \int_0^{\infty} dt' e^{-\lambda t'} \overline{w(t - t') w(t)}.$$

For every value chosen for t' the mean value $\overline{w(t - t') w(t)}$ occurring under the integral sign is taken over the plane PP; this means it refers to all the elements of volume crossing the plane PP at the instant t . However, in the case of stationary turbulent motion, this mean value cannot differ from the mean value of $\overline{w(t - t') w(t)}$ calculated for the history of a single element of volume; that is, calculated as a time mean value by giving a series of values to t , keeping t' constant. Hence this mean value is equal to the Lagrangian correlation for the motion of an element of volume; consequently we shall write R_w for it. In this way the expression for the transportation of particles through the plane PP per unit area and in unit time can be written:

$$(5a) \quad \overline{nw} = - \frac{d\bar{n}}{dz} \int_0^{\infty} dt' e^{-\lambda t'} R_w(t').$$

This can be brought into the form:

$$(5b) \quad \overline{nw} = - D_p \frac{d\bar{n}}{dz},$$

where, written somewhat more accurately:

$$(5c) \quad D_p = \int_0^{\infty} dt' e^{-\lambda t'} w(t - t') w(t) .$$

This quantity D_p is the "turbulent diffusion coefficient." In defining the average value , we have now averaged also over various values of the exponential factor, since elements of volume can have different sizes and will have different exchange coefficients λ . It will be evident that when the rate of exchange of particles between neighboring elements on the whole is slow, so that all λ are small, D_p will practically be given by the ordinary integral of the Lagrangian correlation function, as considered before in section 10. When the rate of exchange is large, the turbulent diffusion coefficient will be smaller provided the correlation curve is of simple type (we come back to this point in section 37). These results confirm those deduced from the general reasoning of section 28.

The expression (5c) for D_p is also applicable when, instead of particles carried by the fluid, we consider a dissolved substance. Instead of the particle density n , we then better consider the concentration c (defined as mass per unit volume) of the dissolved substance. In this case probably λ will mainly depend on true molecular diffusion.

32. The complete expression for the strength of the current of particles, combining the transport due to gravity with that due to the random motions, is:

$$(6) \quad M = -V_s \bar{n} - D_p \frac{d\bar{n}}{dz} .$$

In the case of a stationary field M must have a constant value (independent of z), which will be zero if the boundaries of the field cannot be penetrated by the particles.

When the value is not constant, the field cannot be stationary. The deduction given above loses somewhat of its applicability, but for slow rates of change . in a field of homogeneous turbulence, we can use

the equation:

$$(7) \quad \frac{\partial \bar{n}}{\partial t} = - \frac{\partial M}{\partial z} = v_s \frac{\partial \bar{n}}{\partial z} + D_p \frac{\partial^2 \bar{n}}{\partial z^2} .$$

If the turbulence is not homogeneous, the definition of the correlation function $\overline{w(t - t') w(t)}$ will require adjustment and may differ from the Lagrangian correlation. Its value will become a function of the level \underline{z} for which M must be obtained. Equation (7) is replaced by:

$$(8) \quad \frac{\partial \bar{n}}{\partial t} = v_s \frac{\partial \bar{n}}{\partial z} + \frac{\partial}{\partial z} \left(D_p \frac{\partial \bar{n}}{\partial z} \right) .$$

33. Reaction of the Particles on the Motion of the Fluid. We have assumed in the preceding calculations that gravity gives a constant velocity of fall V_g to the particles (basing ourselves on the equations of motion for spherical particles at low Reynolds numbers). This means that the extra weight of particles, above that of the fluid displaced by them, is balanced by the average resistance they experience from the fluid. In turn they exercise a downward force on the fluid, equal to the excess of weight.

We can make use of this result in writing down an equation of motion for an element of volume of the fluid. Instead of referring to the complete hydrodynamical equations, we shall use the following simplified form:

$$(9) \quad \rho \frac{dw}{dt} = - \frac{d\bar{p}}{dz} + F_z - g\rho - \kappa \rho w$$

On the right hand side the first term represents the mean pressure gradient, which is connected with the mean density by the relation:

$- d\bar{p}/dz = g\bar{\rho}$. The term F_z has been written for the effect of randomly changing pressures around the element connected with the turbulent motion of the field. (This term may include random effects of friction.) Then comes the action of gravity on the element we are considering, $- g\rho$. Finally the term $-\kappa \rho w$ has been introduced as a measure for the average resistance experienced by the element in its motion between the surrounding elements. This expression is no more than an approximation. In so far as the resistance depends on viscosity, a formula might be derived from the viscosity terms in the Navier-Stokes equations which would lead to a factor proportional to the kinematical viscosity of the fluid and roughly inversely proportional to the square of the diameter of the element.

The density ρ of an element depends on the number of particles per unit volume contained in it, or on the mass concentration of suspended or dissolved material. We assume that there is a linear relation between concentration and density so that our previous equation (4) can be replaced by:

$$(4a) \quad \rho - \bar{\rho} = - \frac{d\bar{\rho}}{dz} \int_0^{\infty} dt' e^{-\lambda t'} w(t - t').$$

This can be used to eliminate the group of terms:

$$- \bar{dp}/dz - g\rho = -g(\rho - \bar{\rho})$$

from eq. (9). Since we then have taken sufficient account of the density fluctuations, we can now replace ρ by $\bar{\rho}$ in the terms $\rho(dw/dt)$ and $-\kappa \rho w^*$.

In this way we arrive at an integro-differential equation for w . By means of a simple transformation it can be reduced to the following ordinary differential equation of the second order:

$$(10) \quad \frac{d^2 w}{dt^2} + (\kappa + \lambda) \frac{dw}{dt} + (\beta g + \kappa \lambda) w = \frac{1}{\bar{\rho}} \left(\frac{dF_z}{dt} + \lambda F_z \right).$$

Here the letter β has been used for

$$(10a) \quad \dots \beta = - \frac{1}{\bar{\rho}} \frac{d\bar{\rho}}{dz} = - \frac{d(\mathcal{L} n \bar{\rho})}{dz}.$$

34.** We first discard the effect of the forces F_z and consider the homogeneous equation:

$$(10b) \quad \frac{d^2 w}{dt^2} + (\kappa + \lambda) \frac{dw}{dt} + (\beta g + \kappa \lambda) w = 0.$$

If $d\bar{\rho}/dz = 0$, which makes $\beta = 0$, that is in the absence of any density gradient, the solutions of this equation will represent damped motions. A positive value of $d\bar{\rho}/dz$ makes β negative. When $\beta g + \kappa \lambda < 0$, solutions appear which represent motions increasing in time. This means that a situation in which the density increases upwards (heavier layers on top of lighter layers) will be unstable if the gradient exceeds a certain limit. With a negative density gradient (lighter layers above the heavier ones)

*In a fluid heavily loaded with sediment (or with a dissolved substance), the increase in density can materially affect the inertia of the elements. The simplification introduced in the text then is not allowed.

**In preparing this section and sections 36 - 39, great help has been derived from a report by ir. J.C. Schönfeld of the "Rykswaterstaat" (Government Water Board) in Holland, written for a seminar on turbulence at the Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool, held in the spring of 1950.

all solutions of (10b) are damped; hence situations with negative $d\bar{\rho}/dz$ are always stable. In the case

$$\beta g > \frac{(\kappa - \lambda)^2}{4}$$

the solutions will have a periodic character.

When the right hand side is re-introduced, the appropriate solution of (10) for the last mentioned case can be written in the form:

$$(11) \quad w = \frac{1}{n} \int_0^{\infty} dt' e^{-kt'} \sin nt' \varphi(t - t'),$$

$$\text{where } k = \frac{1}{2}(\kappa + \lambda); \quad n^2 = \beta g - \frac{(\kappa - \lambda)^2}{4};$$

while φ has been used as an abbreviation for

$$\varphi = \frac{1}{\bar{\rho}} \left(\frac{dF_z}{dt} + \lambda F_z \right).$$

Much now depends on the magnitude of κ and λ . If the resistance, expressed by the coefficient κ , would be due mainly to ordinary viscosity, this coefficient appears to be of the order of magnitude $3\nu/R^2$, ν being the kinematic viscosity of the fluid and R a mean radius of the element. If the exchange of the transported material is exclusively due to ordinary molecular diffusion, it is generally found that λ is much smaller than κ . When effects of small scale motions must be taken into account, the value is difficult to estimate; but it will be seen already, how much depends on the size of the elements of volume which can be considered as moving more or less individually. (With large elements, complications present themselves which have been left out of sight in the preceding deductions; for instance, exchange of particles or of dissolved substance will mainly be limited to surface layers; further, in determining the transport of material through the plane PP, considered before, we must give attention to the circumstance whether the center of the element is in this plane or is at an appreciable distance above or below it.)

If there is no pre-existing turbulence, but only a stratified system in which the forces indicated by F_z would have the character of

minor disturbing effects, a positive value of $d\bar{\rho}/dz$, will always bring instability. The dimensions of the elements of volume may become large, depending on features of the distribution of the small disturbances z . In the case of a negative $d\bar{\rho}/dz$, all motions will be damped, but there may appear motions of periodic character with very small damping.

If turbulence is already existing, the dimensions of the randomly moving elements of volume will be mainly determined by the character of the turbulence. The transport phenomena can then be considered as secondary effects. Nevertheless, the circumstance that eq. (10b) leads to damped motions when $d\bar{\rho}/dz < 0$ indicates that loss of energy is connected with the transport phenomena. This can also be seen from eq. (9) if this equation is multiplied by \underline{w} and a mean value is taken over all elements crossing the plane PP; we then obtain

$$(12) \quad \frac{d}{dt} \left(\frac{1}{2} \bar{\rho} \bar{w}^2 \right) = \overline{F_z w} - g \overline{\rho w} - \kappa \bar{\rho} \bar{w}^2.$$

The left hand side describes the increase of the average kinetic energy of the vertical motion. The mean value $-\bar{w} (d\bar{\rho}/dz)$, in which $d\bar{\rho}/dz$ is a constant independent of w , drops out. The term $\overline{F_z w}$ represents the average expenditure of energy by the fluctuating pressure gradients maintaining the vertical random motion. This energy must be derived from other forms of motion, a question to which we shall give some attention later on, in connection with the problem of exchange of momentum. At the present moment this quantity may be considered as given. The last term of the equation, $\kappa \bar{\rho} \bar{w}^2$, represents a loss of energy through friction; this is part of the ordinary dissipation always to be found in turbulent motion. Finally the term $-g \overline{\rho w}$ is the only one in which we take into account the correlation between ρ and w . According to (4) and (5) it can be written:

$$(12a) \quad -g \overline{\rho w} = g \frac{d\bar{\rho}}{dz} \int_0^{\infty} dt' e^{-\lambda t'} \overline{w(t-t') w(t)} = -\beta g \bar{\rho} D_p.$$

For negative $d\bar{\rho}/dz$, so that $\beta > 0$, this term gives the extra loss of energy connected with the turbulent spreading of a heavy admixture. According to (12a) the extra loss of energy should be less for elements

of very small dimensions, since for them λ will be large, which reduces the integral.

It has been observed that a large negative value of $d\bar{\rho}/dz$ may favor the appearance of a more or less periodic motion of elements of sufficient size. If such motions appear, they may influence the form of the correlation curve. We come back to this in section 37.

35. Transport of Heat. The transport of heat depends on turbulence in much the same way as the transport of particles, etc., but certain details require separate consideration.

In the first place, temperature changes produce changes in density and thus affect the volume of the randomly moving elements. Although the change of volume is small, account of it must be taken in calculating the amount of work done in transporting elements of volume in a field with a temperature gradient. We, therefore, introduce the density ρ of the element, and obtain an aggregate transport of mass through our plane PP: $\sum \omega_i \rho_i w_i = \overline{\rho w}$. In the present case, although the elements of volume may exchange heat and thus change their density, we can assume that there is no resulting transport of matter. This has the consequence that the condition $\bar{w} = 0$, which we could use for incompressible motion, must now be replaced by $\overline{\rho w} = 0$.

If the temperature of an element of volume is T , the transport of heat, per unit area and in unit time, is given by:

$$\sum \omega_i \rho_i w_i C_v T_i = C_v \overline{\rho w T},$$

C_v being the specific heat at constant volume. At the same time, work is done by the pressure to the amount:

$$\sum p_i \omega_i w_i = R \sum \omega_i \rho_i w_i T_i = R \overline{\rho w T},$$

R here being the gas constant, so that $p = R\rho T$. Hence the total transport of energy is given by:

$$(13) \quad Q = C_p \overline{\rho w T},$$

where $C_p = C_v + R$ is the specific heat at constant pressure.

Further, the temperature of an element of volume, during its random movement, does not only change through conduction but also in consequence of expansion or contraction as it comes into regions of different pressure. There will be a systematic effect connected with the mean pressure gradient in the field, which itself is connected with gravity. Applying Poisson's law, we find a rate of change of temperature for an element of volume possessing the vertical velocity \underline{w} :

$$\left(\frac{dT}{dt} \right)_{\text{Poisson}} = \frac{\gamma - 1}{\gamma} \frac{\bar{T}}{\bar{p}} w \frac{d\bar{p}}{dz} = - \frac{\gamma - 1}{\gamma} w g \frac{\bar{T}}{\bar{p}} - \Gamma w .$$

Here $\gamma = C_p/C_v$, while $\Gamma = \frac{\gamma - 1}{\gamma} g$ is the so-called adiabatic lapse rate of the temperature. It should be noted that if we had investigated heat transport in a horizontal direction, in a field which does not show a mean pressure gradient in the direction we are considering, Γ must be replaced by zero. - Adding this to the rate of change by conduction of heat from an element to its surroundings, or inversely, for which we again will make use of a coefficient λ , we obtain:

$$(14) \quad \frac{dT}{dt} = \lambda (\bar{T} - T) - \Gamma w + \psi .$$

An extra term ψ has been introduced, which can be used if there are other effects influencing the temperature, for instance radiation or condensation phenomena. When there is no need to take such effects into account, ψ can be omitted.

If we use the letter b now to denote the mean temperature gradient \bar{dT}/dz , we can write:

$$\bar{T}(t - t') = (C + b z_p) - b \int_0^{t'} w(t - t'') dt''$$

for the mean temperature at the level where our element of volume found itself at the instant $t - t'$. With the aid of this formula, the integral of (14) can be brought into the form:

$$(15) \quad T - \bar{T} = -(b + \Gamma) \int_0^\infty dt' e^{-\lambda t'} w(t - t') + \int_0^\infty d\lambda' e^{-\lambda' t'} \psi(t - t'),$$

\bar{T} here denoting the mean temperature at the level where the element finds itself at the instant t , i.e., at the plane PP. The transport of energy across this plane, per unit area and in unit time, now becomes:

$$(16) \quad Q = -(b + \Gamma) c_p \bar{\rho} \int_0^{\infty} dt' e^{-\lambda t'} w(t-t') w(t) + c_p \bar{\rho} \int_0^{\infty} dt' e^{-\lambda t'} \psi(t-t') w(t).$$

In the expressions for the averages we have taken apart the mean density $\bar{\rho}$ since the inaccuracy introduced in this way can be neglected.

If we leave aside the effects included under the term ψ , it will be seen that the heat transport in the vertical direction depends on the factor $b + \Gamma$. It vanishes when the mean temperature gradient $b = d\bar{T}/dz = -\Gamma$, that is, when the temperature decreases according to the adiabatic lapse rate. The so-called potential temperature then is constant in the vertical direction.

An interesting feature may be connected with the term ψ . Since the temperature of the element influences its density and thus its buoyancy with respect to the surrounding elements, ψ can have influence on the motion and a correlation between ψ and w is possible. This can have the effect that $b + \Gamma$ sometimes must have a small positive value to make Q vanish.

36. Transfer of Momentum in a Stratified Turbulent Flow. Until thus far the random motions we have been considering were in the vertical direction. Reference to turbulent motions in other directions has been made only in so far as the pressure force F_z in eq. (9) might depend on them.

If we have to do with a stratified turbulent flow, in which the mean velocity of flow is a function of z , the vertical motion of the elements of volume will bring about a transfer of horizontal momentum. In principle, similar relations are effective in this phenomenon as have been considered in the preceding sections. The quantity transported is the horizontal momentum component (in the direction of the mean flow). We, therefore, need an equation describing how the horizontal velocity is influenced by the motion of the surrounding elements. For this we use:

$$(17) \quad \bar{\rho} \frac{du}{dt} = F_x - K \bar{\rho} (u - \bar{u}) .$$

In this equation, where attention is not directed to the transportation of foreign material or of heat, the mean density can be used. The term F_x , which is similar to F_z occurring in eq. (9), represents the effect of pressures giving a resultant in the horizontal direction, while K again has been used to describe the resistance experienced by the horizontal motion. The mean velocity of flow in the horizontal direction \bar{u} (often written U) is a function of z .

If we write $d\bar{u}/dz = U'$, the integral of (17) becomes:

$$u - \bar{u} = -U' \int_0^{\infty} dt' e^{-Kt'} w(t-t') + \frac{1}{\bar{\rho}} \int_0^{\infty} dt' e^{-Kt'} F_x(t-t') .$$

The transfer of momentum is now given by:

$$(18) \quad \bar{\rho} \overline{uw} = -\bar{\rho} U' \int_0^{\infty} dt' e^{-Kt'} w(t-t') w(t) + \int_0^{\infty} dt' e^{-Kt'} F_x(t-t') w(t) .$$

It is generally assumed that there is no correlation between w and F_x and the last term can be left out .

The transfer of momentum can be described as a shearing stress acting on the mean field of flow. The stress is ordinarily defined with the opposite sign, so that:

$$(18a) \quad \tau_{xz} = -\bar{\rho} \overline{uw} = \bar{\rho} U' \int_0^{\infty} dt' e^{-Kt'} w(t-t') w(t) .$$

37, We have now obtained three transfer coefficients:

$$(I) \quad D_p = \int_0^{\infty} dt' e^{-\lambda t'} w(t-t') w(t)$$

[compare eq. (5c)] for particles, with λ referring to particle exchange between neighboring elements;

$$(II) \dots \bar{\rho} c_p D_q = c_p \bar{\rho} \int_0^{\infty} dt' e^{-\lambda t'} w(t - t') w(t)$$

[compare eq. (16)] for heat, with λ referring to heat conduction between neighboring elements;

$$(III) \dots \rho D_m = \bar{\rho} \int_0^{\infty} dt' e^{-K t'} w(t - t') w(t)$$

[compare eq. (18)] for momentum K referring to the resistance experienced by an element in its horizontal motion. All expressions depend on the same correlation in the vertical motion of the elements, but the exponential factors are different. If the exponential factor is omitted altogether, we arrive at

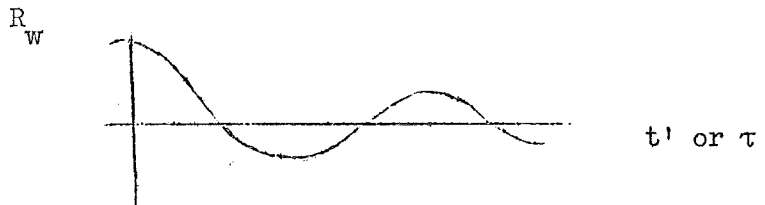
$$(IV) \dots D = \int_0^{\infty} dt' w(t - t') w(t) ,$$

which is equal to the coefficient of turbulent spreading for particles completely following the motion of the elements of volume, as discussed in section 10 (Chapter II).

Comparing D_m and D_p , if we can assume $K > \lambda$, as is the case when the exchange of material between elements of volume is due to molecular diffusion only, one can expect that in general

$$D_m < D_p ,$$

provided the correlation curve is of simple shape (R_w decreasing from its maximum to zero without change of sign). However, if, in the case of a large density gradient there are an important number of elements of volume with a (damped) periodic motion, there seems to be a possibility for R_w to be of the form:



In such a case it is possible that

$$D_m > D_p$$

for particular values of K , even if $K > \lambda$.

Measurements by J.P. Jacobsen in the Danish Waters (Rapports et Proc. Verbaux des Réunions du Conseil Permanent pour l'Exploration de la Mer, vol. 64, p. 59, 1930) have shown that such cases are found in nature, although cases with $D_m < D_p$ are found more commonly.

The investigations made by Dr. V. Vanoni, "Transportation of Suspended Sediment in Water", Trans. Amer. Soc. Civil Engineers, 111, p. 67-133, 1946, likewise have given cases where D_p can be either larger or smaller than D_m . It was found that for fine material D_p tended to be larger than D_m , which is in accordance with the first case mentioned above, so that we are led to assume that with fine sediment $K > \lambda$. With coarser sediment the opposite relation was obtained. I would suppose that in this case the explanation must not be sought in the appearance of periodic motions, but rather in an increased tendency of coarse particles to escape from elements of volume to neighboring ones, which might make the factor λ for them to be much larger than the coefficient K .

38. When a shearing stress τ_{xz} acts on a field of flow with a mean velocity gradient $du/dz = U'$, the energy transmitted to the turbulent motions per unit volume and in unit time is:

$$(19) \quad U' \tau_{xz} = \bar{\rho} D_m (U')^2$$

This energy is derived from the work done by the exterior forces driving the fluid, for instance in the case of flow through a tube from the longitudinal pressure drop, and in the case of an inclined canal from gravity. It is finally spent through friction in the turbulent motion. If there is a negative density gradient, work will also be spent as a consequence of turbulent mixing. According to (12a), this latter work is given by: $\beta g \bar{\rho} D_p$. Hence we may write as a general expression for the energy balance:

$$(20) \quad U' \tau_{xz} = \bar{\rho} D_m (U')^2 = \beta g \bar{\rho} D_p + \text{work lost through viscous friction. Since the work lost through viscous friction necessarily is}$$

positive, we must have:

$$(21) \quad D_m/D_p > r,$$

where r has been written for the dimensionless parameter:

$$(21a) \quad r = \beta g / (U')^2$$

which was first introduced by L.F. Richardson, Proc. Roy. Soc. London, A, 97, p. 354, 1930. If r is too large, so that this inequality cannot be satisfied, turbulence will be impossible; this again explains the stabilizing effect of large negative density gradients.

The inequality (21) has been given in that form by G.I. Taylor (Rapp. et Proc. Verb. des Réunions du Conseil Permanent Intern. pour l'Exploration de la Mer, Vol. 76, p. 35, 1931). In former work it had been tacitly assumed that D_p and D_m would always be equal, so that the critical value of r would be unity. Measurements by J.P. Jacobsen proved that values of r much above unity are well compatible with turbulence. From simultaneous measurements of the velocity and the salt distribution, data could be obtained making possible the calculation of D_m and D_p , and which showed that on the whole the relation (21) is satisfied. (Compare e.g., S. Goldstein, Recent Developments in Fluid Dynamics, Oxford, 1938, Vol. I, p. 229-232).

39. As regards the work spent in viscous friction, let us still write down an approximate equation for the movements in the transverse direction, which thus far have not been considered:

$$(22) \quad \bar{\rho} \frac{dv}{dt} = F_y - K \bar{\rho} v$$

This equation has been constructed on the same principles as those used for u and w .

For a stationary turbulent field, we can ~~now~~ write down the energy balance in the following way:

$$\mu_1 = u - \bar{u}$$

$$(23) \quad \bar{\rho} D_m (U')^2 = \beta g \bar{\rho} D_p - (\overline{F_x u} + \overline{F_y v} + \overline{F_z w}) + \bar{\rho} K (\overline{u^2} + \overline{v^2} + \overline{w^2})$$

The pressure forces F_x , F_y , F_z will be coupled. It can be expected that they only serve to transmit energy from one type of motion

(say from the horizontal motion) to the other types (motions in the directions of y and z). We may expect, therefore, that

$$\overline{F_x u} + \overline{F_y v} + \overline{F_z w} = 0$$

The energy balance then reduces to:

$$(24) \quad \bar{\rho} D_m (U')^2 = \beta g \bar{\rho} D_p + \bar{\rho} \kappa (\overline{u^2} + \overline{v^2} + \overline{w^2}) .$$

It expresses that all energy derived from the main motion with its velocity gradient U' , is spent, partly in overcoming gravity (in so far as foreign material is carried), partly in viscous friction.

40. Note on the Concept of Mixing Length. - The expressions for D_p , D_q and D_m considered in section 37 have the dimensions $(\text{velocity})^2 \cdot (\text{time})$, which is equivalent to $(\text{velocity}) \cdot (\text{length})$. In many considerations on the processes of exchange of momentum and of mixing, they are replaced by the mean value of the product of the velocity component $w(t)$ [the velocity of the element of volume when it crosses the level PP] into a length ℓ , which is considered to represent the distance travelled by the element since the instant when it made itself free of its surroundings for the last time before coming to PP. This distance is often called the "free path," in reminiscence of a similar quantity occurring in the kinetic theory of gases. It is then assumed that the element brings with it the value of the mean horizontal velocity, or the mean concentration of matter, or the mean temperature as found at the level $z_p - \ell$.

Formally there is no objection to writing, e.g.:

$$\ell = \int_0^{\infty} dt' e^{-\kappa t'} w(t - t')$$

which makes it possible to write

$$D_m = \overline{\ell w} .$$

But this way of writing somewhat obscures the circumstance that the exponential function in the expressions for D_p , D_q and D_m will be different. The concept of an element of volume, making itself completely free out of surroundings in which until that instant it had been caught, is more

crude and is less adaptable than the idea of a gradual exchange. In mixture length theory very often the same length is used for all cases and, consequently, it is often assumed that the three transfer coefficients have the same value. This would make it possible, for instance, to make calculations on heat transfer when the magnitude of the momentum transfer could be obtained from considerations on the force equilibrium. Difficulties, however, have been encountered in interpreting certain observational results concerning simultaneous momentum and heat transfer. This has led to an investigation into the problem whether in some cases instead of transfer of momentum (that is, of velocity) one should not rather consider transfer of vorticity. We come back to this point in section 47.

Chapter VI

FEATURES OF THE NAVIER-STOKES EQUATIONS

41. The Reynolds' Stresses. What we have done in the preceding chapter can be described as the calculation of certain Eulerian mean values from the Lagrangian correlation for the velocity of an element of volume. These Eulerian mean values:

\overline{nw} (eq. 5); \overline{pw} (eq. 12a); \overline{pwt} (eqs. 13 and 16); \overline{uw} (eqs. 18 or 18a)

has been defined as mean values taken over a plane PP, at a definite instant of time. In a stationary field they will be independent of the time and can also be defined as time mean values taken at a given point. The calculation was possible in consequence of the particular situation we had considered: in the field there was present a certain gradient either of concentration, or of temperature, or of horizontal velocity, which gradient was maintained all the time; we could then find out how elements of volume during their wanderings over this field, pick up either foreign matter, or heat, or momentum.

The Eulerian mean values come to the foreground when we apply the Navier-Stokes equations to deduce laws for the main flow. In the present chapter we will investigate some of the features of these equations. We restrict to momentum transfer, since this forms the most important problem.

The Navier-Stokes equations for the motion of a fluid of constant density have the following form:

$$(1) \dots \rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = - \frac{\partial p}{\partial x_i} + \rho \nu \Delta u_i$$

$$(2) \dots \frac{\partial u_i}{\partial x_i} = 0$$

For convenience the coordinates have been denoted by x_1, x_2, x_3 ; the components of the velocity by u_1, u_2, u_3 ; where repeated indices occur,

it is understood that a summation is carried out.

In virtue of the equation of continuity, it is possible to write:

$$u_j \frac{\partial u_i}{\partial x_j} = - \frac{\partial}{\partial x_j} (u_i u_j).$$

Equations (1) consequently can be transformed into:

$$(1a) \dots \rho \frac{\partial u_i}{\partial t} = - \frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_j} (\rho u_i u_j) + \rho \nu \Delta u_i$$

Sometimes the original form of the non-linear terms is more convenient; in other cases the new form has advantages.

We will assume that the field is stationary in the statistical sense, so that time mean values will exist for any variable quantity. The mean values of the velocity components will be denoted by U_1, U_2, U_3 ; that of the pressure by P . These four quantities are independent of the time. If the fluctuating or turbulent parts of the velocity components and of the pressure are denoted by u'_1, u'_2, u'_3, p' , we shall have:

$$u_i = U_i + u'_i ; \quad p = P + p' .$$

These expressions can be substituted into eqs. (1) or (1a) and (2), which lead to the following results (for simplicity the accents have been omitted after the substitution, so that all small letters now indicate turbulent quantities):

$$\begin{aligned} \rho \left(\frac{\partial u_i}{\partial t} + U_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial U_i}{\partial x_j} + u_j \frac{\partial u_i}{\partial x_j} \right) = \\ = - \frac{\partial P}{\partial x_i} - \frac{\partial p}{\partial x_i} + \rho \nu \Delta U_i + \rho \nu \Delta u_i ; \\ \frac{\partial U_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i} = 0 \end{aligned}$$

We take mean values of all terms occurring in these equations

in order to obtain a set of equations governing the main flow. In this process terms linear in the turbulent components drop out. The terms of the second degree in the turbulent components, however, will not drop out in general. We write:

$$(3) \quad \tau_{ij} = - \rho \overline{u_i u_j}$$

The quantities so obtained are expressions for the momentum transfer due to the turbulent motion. They are analogous to similar expressions used in the kinetic theory of gases for the explanation of the major part of the viscous forces. In hydrodynamics they are known as the Reynolds' stresses.

The equations for the main flow can now be brought into the form:

$$(4) \quad \rho U_j \frac{\partial U_i}{\partial x_j} = - \frac{\partial P}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + \rho \nu \Delta U_i$$

$$(5) \quad \frac{\partial U_i}{\partial x_i} = 0$$

If these equations are subtracted from the full equations, we are left with a set of equations governing the turbulent motion. They are of more complicated type and can be written:

$$(6) \quad \rho \left(\frac{\partial u_i}{\partial t} + U_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial U_i}{\partial x_j} \right) = - \frac{\partial p}{\partial x_i} - \frac{\partial (\rho u_i u_j - \rho \overline{u_i u_j})}{\partial x_j} + \rho \nu \Delta u_i$$

$$(7) \quad \frac{\partial u_i}{\partial x_i} = 0$$

42. From equation (4) it will be seen that the Reynolds' stresses can be considered as a system of interior forces, acting on the main flow in consequence of the presence of the turbulent motion. The component τ_{xz} of these stresses is the one we already encountered in eqs. (18) and (18a) of the preceding chapter.

The equations are greatly simplified if we restrict to the case

where the mean flow is in the direction of the x-axis only. It follows from the equation of continuity that the velocity component U_1 (for which we can simply write U) must be independent of x . We shall assume, moreover, that it is independent of y . We write $dU/dz = U'$; $d^2U/dz^2 = U''$. In this case we can assume that the turbulent motion is not only stationary with respect to time, but that, statistically speaking, it will also be independent of x and y . Hence also the mean values of u, v, w, p calculated with respect to x or with respect to y will be zero. The pressure P will depend linearly on x ; $\partial P/\partial x$ will be a constant throughout the whole field; $\partial P/\partial y = 0$; $\partial P/\partial z$ will be independent of x and y .

Since there is no acceleration of the main motion, the left hand side of eq. (4) becomes zero. Derivatives of the Reynolds' stresses with respect to x and y drop out; and $\tau_{xy} = -\rho \overline{uv}$ and $\tau_{yz} = -\rho \overline{vw}$ both are zero from reasons of symmetry. Hence we are left with:

$$(8) \quad \begin{cases} 0 = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} + \rho v U'' \\ 0 = -\frac{\partial P}{\partial z} + \frac{\partial \tau_{zz}}{\partial z} \end{cases}$$

where $\tau_{xz} = \rho \overline{uw}$ and $\tau_{zz} = -\rho \overline{w^2}$. The second equation is not very important. The first one shows that the Reynolds' stress τ_{xz} balances the combined effect of the average pressure gradient and of the viscous friction experienced by the main flow. Since in a large part of the field this viscous friction is exceedingly small, it will be evident that the Reynolds' stress τ_{xz} is the principal quantity determining the resistance experienced by the main flow.

One of the major aims of turbulence theory is to find a method for calculating this stress directly from the equations governing the turbulent motion (eqs. 6 and 7), taken together with appropriate boundary conditions, but without introducing assumptions on Lagrangian correlations or the like.

At present the theory is not yet developed far enough to make possible the execution of this program. Nevertheless it is of interest

to give some time to an analysis of various features of the equations mentioned, since this will throw light on the character of the turbulent motion. Examples will be given in the next sections.

In the present case equations (6) take the form:

$$\begin{aligned} \rho \left(\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + w U' \right) &= - \frac{\partial p}{\partial x} - \frac{\partial}{\partial x_j} (\rho u_1 u_j - \rho \overline{u_1 u_j}) + \rho \nu \Delta u \\ (9) \quad \rho \left(\frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x} \right) &= - \frac{\partial p}{\partial y} - \frac{\partial}{\partial x_j} (\rho u_2 u_j - \rho \overline{u_2 u_j}) + \rho \nu \Delta v \\ \rho \left(\frac{\partial w}{\partial t} + U \frac{\partial w}{\partial x} \right) &= - \frac{\partial p}{\partial z} - \frac{\partial}{\partial x_j} (\rho u_3 u_j - \rho \overline{u_3 u_j}) + \rho \nu \Delta w \end{aligned}$$

43. The Energy Equations for the Main Flow and for the Turbulent Field.

To obtain the energy equation for the main flow, in the simple case to which refer eqs. (8), we multiply the first equation of (8) by U ; this gives:

$$0 = -U \frac{\partial p}{\partial x} + U \frac{\partial \tau_{xz}}{\partial z} + \rho \nu U U''.$$

The second and third terms on the right hand side can be transformed into:

$$\frac{\partial}{\partial z} \left\{ U (\tau_{xz} + \rho \nu U') \right\} = U' \tau_{xz} + \rho \nu (U')^2,$$

and the equation can be re-written:

$$(10) \quad -U \frac{\partial p}{\partial x} + \frac{\partial}{\partial z} \left\{ U (\tau_{xz} + \rho \nu U') \right\} = U' \tau_{xz} + \rho \nu (U')^2$$

The first term on the left hand side is the energy supplied to the field per unit volume and in unit time through the pressure gradient which maintains the main flow. The second term has the form of a derivative; this implies that it represents a transfer of energy from an

element to adjacent ones. If we do not consider a single element of volume, but integrate the equation over a large domain, bounded by two planes $z = \text{constant}$ at which U is zero, this term disappears. If U is not zero at the bounding planes, there can be transfer of energy to the flow from the outside, or inversely. But a term having the form of a derivative never represents a loss or a gain of energy in the interior of the field.

On the right hand side we have first a term representing work done in connection with the Reynolds' stress τ_{xz} . This same term will turn up on the left hand side of the equation of energy for the turbulent motion. Its meaning is a transfer of energy from the main motion to the turbulent motion. The second term on the right hand side gives the energy dissipated in consequence of the action of viscosity on the main motion.

Hence the equation expresses that energy derived from outward sources is spent, partly in overcoming the Reynolds' stress, partly through viscous dissipation in the main flow. The first part is transferred to the turbulent motion.

In order to obtain the equation of energy for the turbulent motion, we multiply the three equations of the system (9) by the corresponding components u_i and add; the result can be brought into the form:

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \left(\frac{\rho u_i^2}{2} \right) + \rho u w U' = \\ & = - \frac{\partial}{\partial x_i} (\rho u_i) - \frac{\partial}{\partial x_j} \left(\frac{\rho u_i^2 u_j}{2} \right) + u_i \frac{\partial}{\partial x_j} (\rho \overline{u_i u_j}) + \rho \nu u_i \Delta u_i \end{aligned}$$

We take the mean value and obtain:

$$- U' \tau_{xz} = - \frac{\partial}{\partial x_i} (\overline{\rho u_i}) - \frac{\partial}{\partial x_j} \left(\frac{1}{2} \rho \overline{u_i^2 u_j} \right) + \rho \nu \overline{u_i \Delta u_i}.$$

The term $u_i \Delta u_i$ can be transformed in several ways: the one most commonly used is:

$$u_i \Delta u_i = \frac{\partial}{\partial x_j} \left(u_i \frac{\partial u_j}{\partial x_i} + u_i \frac{\partial u_i}{\partial x_j} \right) - \Phi,$$

where

$$(12) \Phi = 2 \left(\frac{\partial u}{\partial x} \right)^2 + 2 \left(\frac{\partial v}{\partial y} \right)^2 + 2 \left(\frac{\partial w}{\partial z} \right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2.$$

Again several groups of terms in the energy equation have obtained the form of derivatives, meaning that these terms represent transfer of energy between adjacent elements of volume. They can be related to exchange of energy with outside sources through the surfaces limiting the field. In the case of motion through a tube with fixed walls and with turbulence independent of x , such an exchange, however, does not take place. If we leave aside these terms, those remaining depend on the function Φ and represent the loss of energy through viscous dissipation in the turbulent motion. We can write the equation in the form:

$$(11a) \quad U \tau_{xz} = \rho \nu \Phi + \text{derivatives}.$$

It can then be interpreted as follows: on the left hand side we have the energy derived from the main motion through the intermediary of the Reynolds' stresses; on the right hand side we have the dissipation through viscosity; finally there are transfer terms which do not play a part in the process when we consider the field as a whole.

44. The function Φ determining the loss of energy through viscosity, is called the dissipation function. It has great importance in turbulence theory. It is possible to split off some further terms having the form of derivatives; in this way we can obtain:

$$(12a) \quad \Phi = \gamma_x^2 + \gamma_y^2 + \gamma_z^2 + 4 \frac{\partial}{\partial x} \left(\nu \frac{\partial u}{\partial y} \right) + \text{etc.},$$

where the γ 's represent the components of the vorticity. In certain cases the latter form may be the most convenient one; in other cases we keep to eq. (12) for Φ .

The fact that the terms deriving from the turbulent pressure occur in the form of derivatives in eq. (11), proves that they only bring about a transfer of energy, but do not lead to actual losses.

The same implies to the group of terms containing the turbulent velocity components to the third power, which were derived from a multiplication of the non-linear terms in the Navier-Stokes equations with the components of the turbulent velocity. It follows that whenever we attempt to construct a simplified system of formulas, which in their energetical relations should be equivalent to the complete system, care must be taken that expressions substituted for turbulent pressures or for non-linear terms satisfy similar conditions.

Although the non-linear terms in the Navier-Stokes equations do not directly represent a loss or gain of energy, they nevertheless play an important part in promoting dissipation in an indirect way. The effect of these terms on the motion is to steepen velocity gradients in certain parts of the field, which brings about a local intensification of the dissipation.

Velocity gradients calculated from the mean amplitude of the turbulent motion and the average dimensions of the most conspicuous forms of eddy motion, are far too small to lead to the dissipation of energy needed to balance the inflow of energy from outward sources. The really important dissipation takes place in narrow regions or layers in which high velocity gradients have been produced by some process of concentration. It is also this process which ultimately determines the magnitude of the turbulent velocity components.

45. In order to obtain a picture of the way in which narrow regions of dissipation may be produced, we consider a simplified system of equations of motion. We start from the equations for v and w (u_2, u_3);

$$\rho \left(\frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = - \frac{\partial p}{\partial y} + \rho \nu \Delta v$$

$$\rho \left(\frac{\partial w}{\partial t} + U \frac{\partial w}{\partial x} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = - \frac{\partial p}{\partial z} - \frac{\tau_{zz}}{\rho \nu} + \rho \nu \Delta w$$

In these equations we shall neglect the circumstance that U is a function of z and treat U as a constant; we can then introduce a moving coordinate system and take up the derivatives $U(\partial v/\partial x)$, $U(\partial w/\partial x)$ into the time derivative. We further neglect the terms $u(\partial v/\partial x)$, $u(\partial w/\partial x)$; $\nu(\partial^2 v/\partial x^2)$, $\nu(\partial^2 w/\partial x^2)$, partly in connection with the error already introduced by taking U constant, partly on the assumption that on the whole derivatives with respect to x will be of a smaller order of magnitude than derivatives with respect to y or z . Since now u no longer occurs in the equations, we need not introduce the equation of continuity, and we omit all restrictions on the value of $\partial v/\partial y + \partial w/\partial z$. In connection with this we shall neglect the pressure terms. We are then left with:

$$(13) \begin{cases} \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = \nu \left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \\ \frac{\partial w}{\partial t} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = \nu \left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) \end{cases}$$

So long as no steep gradients of v and w have appeared, we moreover can neglect the viscosity and retain:

$$(13a) \quad \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = 0$$

$$\frac{\partial w}{\partial t} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = 0$$

Although the systems (13) and (13a) represent a very mutilated

form of the Navier-Stokes equations, they can serve to demonstrate the tendency for steepening of velocity gradients. The system (13a) possesses characteristics determined by the relations:

$$(14) \quad dy/dt = V; \quad dz/dt = W.$$

Along a characteristic we have:

$$(15) \quad dv/dt = 0; \quad dw/dt = 0,$$

which means that the velocity components are propagated along the characteristics without change. This is, of course, a consequence of the neglect of pressure effects. It follows that the characteristics must be straight lines.

In principle this result makes it possible to derive any subsequent state of the field from an initially given state. Difficulties, however, arise when characteristics meet each other. Without going into the theory we can expect that there will be a marked difference between regions where the motion is divergent and regions where it is convergent. In regions where there is convergence, velocities of different magnitudes and direction will be brought close together, so that steep gradients will be produced.

An analysis of this process can be made when we assume that the initial state of the field is formed by a patchwork of domains, in each of which v and w are linear functions of y and z . It is then found that regions of convergence usually contract into a segment of a straight line, which line becomes the seat of a discontinuity of the velocity, in general both for the normal and for the tangential component (see J. M. Burgers, Proc. Acad. Sciences, Amsterdam 53, p. 122, 1950).

This result, however, requires correction in two ways. In the firstplace, keeping to eqs. (13), as soon as steep gradients appear, the viscosity terms must be taken into account. They prevent the appearance of mathematical discontinuities and turn them into transition layers, with large, but finite gradients. If no other effects are taken

into account, it is found that the dissipation in these transition layers obtains a finite value, independent of the magnitude of the viscosity and given by certain expressions of the third degree in the velocity differences across the layer. This is the main feature in the influence of non-linear terms on the dissipation of energy.

The other correction is necessary because we have neglected the equation of continuity. Convergence of flow in the y, z - plane towards a narrow region requires a large positive value of $\partial u / \partial x$; such a value cannot appear without affecting the pressure distribution, and the latter will react on the breadth of the region of convergence. Since in regions of divergence steep gradients do not appear, only relatively small negative values of $\partial u / \partial x$ are required by these regions and no marked pressure effect is to be expected in them. A case in which the pressure distribution is taken into account will be given in section 47.

There is a further question to be investigated: what happens to a layer of convergence after it has been formed? In general such a layer will not remain at a fixed position of the field; it will be displaced and usually it will obtain a curved form. It is possible that different layers meet each other; there is a probability that they will flow together in such a case.

Some of these points can be investigated by making a still further reaching simplification, in which we restrict to one coordinate and one component of velocity, to which we will come later on.

46. Even without further refinement, the result we have arrived at is of interest in connection with experimental evidence.

The most important point is the difference existing between extension and compression of regions of the field. These two phenomena, although each other's opposite, in their incipient stages, present a markedly different character in their further development. The results of an extension on the whole are not reversible.

One consequence is the tendency of a turbulent field to divide itself into a number of separate regions separated from each other by thin transition layers formed through convergence of the flow in the

y, z- plane. This is of importance in connection with the observational result, that in turbulence elements of volume with different values of the velocity seem to follow each other in an irregular way, with very thin transition layers separating them, as had been mentioned in section 20.

When we do not restrict to the consideration of the flow in a y, z - plane, but give attention to the x- coordinate, the line segments towards which the convergence occurs will appear as the section of flat ribbons extending in the direction of the main flow. We can expect that these ribbons will curl about the streamlines of the main flow. They may start at some place and may coalesce with other ribbons, or may, presumably also disappear at some place further downstream.

Such ribbon-like features are often observed when foreign matter is brought into the flow. An interesting example is to be seen when a strong wind blows over a sandy plane or beach. The sand taken up by the wind moves in thin layers or ribbons, constantly shuffling to and fro, and folding and curling about the streamlines. (Compare also: R.A. Bagnold, "The Physics of Blown Sand and Desert Dunes", London 1941, p. 176-179.) Similar ribbons can sometimes be seen in experiments with water channels. Flames, the curling veils or ribbons of smoke which rise from a lighted cigarette, and veils of vapour rising from a hot liquid belong to the same class of phenomena, since in all of them a certain convergence of the flow plays a part.

There exists a problem whether, in the type of turbulent motion we are considering, vorticity with the axis of rotation more or less parallel to the direction of the main flow can be preponderant over vorticity with axes of rotation directed transversely. [Since we have assumed that U was a function of z , there will be, of course, in the field a steady vorticity depending on dU/dz .]

The question of the preponderance of longitudinal over transverse vorticity has been raised in connection with certain theories of turbulent motion. It has been pointed out, e.g., by S. Goldstein (see "Modern Developments in Fluid Dynamics", Oxford 1938, Vol. I, p. 206-213), that vortex motion with the axis of rotation parallel or perpendicu-

parallel to the direction of the main flow is responsible for the difference between the so-called "momentum transfer" theory proposed by Prandtl to explain the mechanism underlying turbulent friction; and "vorticity transfer", which was the mechanism considered by G. I. Taylor. Whereas vorticity transfer seems to occur in cases where vortex motion with axes perpendicular to the main flow is produced very intensively, as is the case in the flow along a long cylindrical obstacle transverse to the main stream, momentum transfer by "longitudinal" vortices appears to be a feature governing boundary layer flow.

We may add that longitudinal vortices appear always along the walls of tubes or canals when an obstacle is present or when there is a bend in the tube or canal. In these cases the flow pattern has a relatively stable form and leads to what is called "secondary flow"; this does not properly constitute turbulence, but the two types of flow seem to be closely connected. Apparently there is always a tendency to form longitudinal vortices in the neighborhood of walls; when the wall is smooth, these vortices do not have a stable position but will constantly waver about, thus constituting part of the turbulence; when there are certain obstacles, or the like, which stabilize these vortices, they appear as secondary flow. This point of view may bring into connection, for instance, the various opinions brought forward in the discussion of Dr. V. Vanoni's paper on "Transportation of Sediment by Water" (already quoted), Trans. Amer. Soc. Civil Engin. 111, pp. 67-133, 1946.

47. The Increase of Dissipation Caused by the Concentration of Vorticity.

In the simplified example mentioned in section 45 it was found that in regions where $(\partial v / \partial y + \partial w / \partial z) < 0$, there is a tendency to concentrate vorticity into narrow sheets or ribbons. In that example the effect is obtained in a very marked way, owing to the circumstance that the pressure has been neglected. Attempts to take account of the pressure effects in a simple way so far have not been successful. What evidence could be obtained, rather pointed to a much smaller concentration

of vorticity and to vortex sheets having a thickness proportional to \sqrt{W} . A greater concentration, however, can be obtained if we consider concentration towards a line, such as is found when a vortex tube is extended axially. It was G. I. Taylor who first pointed out that the longitudinal extension of vortex tubes must be the main factor in turbulent dissipation, and who also observed that extension must occur more frequently than shortening, since turbulent motion has a diffusive character, so that elements of volume which originally were neighbors will usually tend to move apart (see: G.I. Taylor, Journ. Aeron. Sciences 4, p. 315, 1937; G.I. Taylor and A.E. Green, Proc. Roy. Society London, A, 158, p. 501, 1937; G.I. Taylor, Proc. Roy. Society London, A, 164, p. 15, 1938; also S. Goldstein, Three-dimensional Vortex Motion in a Viscous Fluid, Philos. Mag., (VII) 30, p. 85, 1940)

It is not difficult to calculate what is the ultimate result that can be obtained when a rectilinear symmetrical vortex is extended longitudinally at a constant rate (J.M. Burgers, Proc. Acad. Sciences Amsterdam 43, p. 11, 1940). We consider an axially symmetric field with velocity components u (parallel to the x -axis, which is the axis of the field); v (tangential) and w (radial). It is assumed that:

$$(16) \dots u = 2 A x; \quad w = \frac{A}{r} r,$$

while v shall be a function of r and t which must be found.

In this case there is only one component of vorticity:

$$\gamma_x = \frac{1}{r} \frac{\partial(rv)}{\partial r}$$

We assume that the pressure is given by:

$$p = - \frac{\rho A^2}{2} (4x^2 + r^2) + \rho \int dr \frac{v^2}{r}.$$

Such a pressure field cannot extend over large distances; the expression should be considered as an approximation over a small region containing

a certain length of the x-axis.

The equations of motion in cylindrical coordinates have been given, e.g., in S. Goldstein, Modern Developments in Fluid Dynamics (Oxford, 1938), vol. I, p. 103-104. It will be found that the expressions for u , w , p given above satisfy the equation of continuity and the equations of motion for the axial and radial directions. There remains the equation of motion for v , which has the form:

$$\frac{\partial v}{\partial t} - Ar \frac{\partial v}{\partial r} - Av = v \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} \right) .$$

We will look for a solution independent of the time, assuming that this may represent a state which is asymptotically approached. The resulting equation can be written:

$$Ar \frac{d(rv)}{dr} + v \left\{ \frac{d^2(rv)}{dr^2} - \frac{1}{r} \frac{d(rv)}{dr} \right\} = 0 ,$$

and has the solution:

$$(17) \dots v = \frac{C}{2\pi r} \left(1 - e^{-Ar^2/2v} \right) ,$$

with:

$$(17a) \dots \gamma_x = \frac{AC}{2\pi v} e^{-Ar^2/2v} ,$$

C being the integration constant. It will be seen that vorticity is to be found only in a narrow cylindrical space surrounding the x-axis; the strength of the vortex (circulation along a curve encircling it) is equal to C .

To find the dissipation we must calculate the dissipation function as given by (12). This requires transformation from the cylindrical

coordinates to rectangular ones. The result appears to be of the form:

$$\Phi = 12A^2 + \frac{c^2}{\pi^2 r^4} + \frac{c^2 A^2}{4\pi^2 \nu} e^{-Ar^2/\nu} + \text{terms containing exponential factors and having factors } \nu^{-1} \text{ or } \nu^0.$$

We must multiply Φ by $\rho \nu$ and integrate over the domain to which we apply our solution. The result contains certain terms depending on the magnitude of this volume, which, however, are multiplied by ν ; and further a term independent of ν :

$$(18) \dots \rho \frac{c^2 A}{4\pi} \cdot (\text{length of the vortex}).$$

Hence the resulting dissipation is practically independent of the viscosity. It depends on the number and the strengths of the vortex tubes, and on the absolute value of the difference of the longitudinal velocities u_{II} , u_I at the ends of the extended part. This is seen if we remember that $u = 2Ax$, so that (18) can be written:

$$(18a) \quad \begin{array}{l} \text{dissipation in} \\ \text{an extended vortex} \\ \text{tube} \end{array} = \rho \frac{(\text{circulation})^2}{8\pi} |u_{II} - u_I|$$

48. Note on the Problem of "Momentum Transfer" Versus "Vorticity Transfer".

We return to the first of eqs. (8) for the main flow. The term $\tau_{xz}/\partial z$ in this equation represents what is left from the more complete expression for the Reynolds' stresses which should be written:

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}$$

or:

$$= \frac{\partial}{\partial x} (\rho \overline{u^2}) - \frac{\partial}{\partial y} (\rho \overline{uv}) - \frac{\partial}{\partial z} (\rho \overline{uw}).$$

Now this group of terms can be transformed in a different way, introducing the components of vorticity, giving

$$-\frac{\partial}{\partial x} \left[\frac{\rho}{2} (\overline{u^2 + v^2 + w^2}) \right] + \rho (\overline{v\gamma_z - w\gamma_y})$$

In the case considered here, where the turbulence is the same for all values of x , the first term drops out, and the influence of the turbulent motion on the pressure gradient of the main motion is given by:

$$(19) \dots \rho (\overline{v\gamma_z - w\gamma_y})$$

In order to obtain an estimate of this quantity, the concept of the "mean free path" has been used (see section 40). For a more complete treatment we refer to S. Goldstein, Modern Developments in Fluid Dynamics, (Oxford, 1938), Vol. I, p. 205-214, and literature quoted there. It is assumed that an element of fluid, passing a point P (coordinates x, y, z) at the instant t , originally has made itself free from surroundings at the point a, b, c , where:

$$a = x - l_1; \quad b = y - l_2; \quad c = z - l_3.$$

When the velocity U of the main motion is a function of z only, this element, when at a, b, c , had the mean vorticity existing there with components:

$$\overline{\gamma_x} = 0; \quad \overline{\gamma_y} = U' - l_3 U''; \quad \overline{\gamma_z} = 0.$$

When the element has arrived at P (x, y, z), it will have taken its vorticity with it, but the vorticity vector may have turned ^{and} have been extended, dependent on the motion of the element. The components of vorticity with which it arrives at P will be given by:

$$\gamma_x = \overline{\gamma_y} \frac{\partial x}{\partial b}; \quad \gamma_y = \overline{\gamma_y} \frac{\partial y}{\partial b}; \quad \gamma_z = \overline{\gamma_y} \frac{\partial z}{\partial b}$$

The expression (19) consequently takes the form:

$$(20) \dots \rho (U' - l_3 U'') (v \frac{\partial z}{\partial b} - w \frac{\partial y}{\partial b})$$

Two particular cases can now be considered separately. First assume that all turbulent motions are confined to the x,z-plane and are independent of y, as would be the case with purely two dimensional turbulence. The only vorticity component possible then is γ_y . We have:

$$\frac{\partial u}{\partial y} = 0, \quad v = 0, \quad \frac{\partial w}{\partial y} = 0$$

from which it follows that:

$$\frac{\partial x}{\partial b} = 0; \quad \frac{\partial y}{\partial b} = 1; \quad \frac{\partial z}{\partial b} = 0$$

The expression (20) consequently changes into:

$$- \rho U' \bar{w} + \rho U'' \overline{l_3 w}$$

The first term drops out, since $\bar{w} = 0$. If we write l for l_3 , there remains:

$$(21) \dots + \rho \frac{d^2 U}{dz^2} \overline{l w}.$$

This expression had been obtained by G.I. Taylor as a result of the "vorticity transfer theory" (1915).

Next, suppose that the turbulent motion is of such a nature that $\partial u / \partial x$, $\partial v / \partial x$ and $\partial w / \partial x$ can be put equal to zero. We then have:

$$\partial v / \partial y + \partial w / \partial z = 0$$

We then have:

from which it follows that: $\partial l_2 / \partial y + \partial l_3 / \partial z = 0$

$$\frac{\partial y}{\partial b} = 1 + \frac{\partial l_2}{\partial y} = 1 - \frac{\partial l_3}{\partial z}; \quad \frac{\partial z}{\partial b} = \frac{\partial l_3}{\partial y}$$

Expression (19) now takes the form:

$$\begin{aligned} & \rho (U' - l_3 U'') \left(v \frac{\partial l_3}{\partial y} - w + w \frac{\partial l_3}{\partial z} \right) = \\ & = \rho U'' l_3 w + \rho U' \left(v \frac{\partial l_3}{\partial y} + w \frac{\partial l_3}{\partial z} \right) - \\ & - \frac{1}{2} \rho U'' \left(\frac{\partial v l_3^2}{\partial y} + \frac{\partial w l_3^2}{\partial z} \right). \end{aligned}$$

Here $v \frac{\partial l_3}{\partial y} = - l_3 \frac{\partial v}{\partial y} = + l_3 \frac{\partial w}{\partial z}$

Further $\frac{\partial}{\partial y} v l_3^2 = 0$; $\frac{\partial}{\partial z} w l_3^2$ is neglected.

In this way there remains:

$$(22) \dots \rho U'' l_3 w + \rho U' \frac{\partial}{\partial z} l_3 w = \rho \frac{d}{dz} \left(\frac{dU}{dz} l_3 w \right)$$

The two assumptions used, that of turbulence being confined to motions in the x,z-plane, and (of that) motion confined to the y,z-plane, are in some way connected with different assumptions about the effect of turbulent pressure fluctuations. Formula (22) is obtained from the theory of momentum transfer, in which it is assumed that an element of volume takes its u-velocity with it over a free path, uninfluenced by pressure fluctuations. In the vorticity transfer theory this assumption is not introduced, but it is supposed that the strength of a vortex with the axis parallel to the y-axis, is unaffected by the transport over the free path.

49. Experimental evidence has shown that in cases where turbulent motion is superposed on a main motion, as in the flow through a tube, the amplitudes of the turbulence usually are a few percent only of the velocity in the main motion. This suggests that perhaps certain characteristic features of turbulence could be deduced from simplified equations of motion in which terms of the second degree in u , v , w have been omitted. It might be possible to analyze the coupling between the main motion and the turbulent motion. The coupling between the various components of turbulent motion themselves will not be obtained in this way; hence the picture certainly will not be applicable at the far end (the small wavelength end) of the spectrum. Here the coupling between the components of turbulent motion is far more important than coupling with the main motion and it is highly probable that at the small wavelength end turbulence assumes a universal pattern, practically independent of the form of the main motion. This pattern is described in the theory of isotropic turbulence to which we will come later.

For the first few long wavelength components the coupling with the main motion probably will be more important than the coupling between these components themselves. However, it is certain that the aggregate effect of the small wavelength components is of great importance. It is this circumstance which makes it difficult to arrive at definite results about the long wavelength components from linearized equations. It has been proposed in certain investigations to represent the aggregate effect of the small wavelength components as an increased viscosity, but the magnitude of the coefficient to be used is not known.

The linearized equations for the turbulence are identical with the equations used in investigations on the stability of laminar flow. They are obtained from eq. (9) of Section 42 and have the form:

$$\begin{aligned}\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + wU' &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \Delta u \\ \frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x} &= -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \Delta v \\ \frac{\partial w}{\partial t} + U \frac{\partial w}{\partial x} &= -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \Delta w \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0\end{aligned}\tag{23}$$

It is assumed that U is a given function of z . Solutions of this system of equations have been derived by putting:

$$\begin{aligned} u &= \phi_1(z) \cdot e^{i(\alpha x + \beta y + \omega t)} \\ v &= \phi_2(z) \cdot e^{\dots} \\ w &= \phi(z) \cdot e^{\dots} \\ p/\rho &= P(z) \cdot e^{\dots} \end{aligned} \tag{24}$$

where ϕ_1 , ϕ_2 , ϕ , P represent unknown functions of z . When these expressions are substituted into (23), the functions ϕ_1 , ϕ_2 , P can be eliminated; the remaining equation for ϕ is:

$$(U - \frac{\omega}{\alpha}) \left\{ \phi'' - (\alpha^2 + \beta^2) \phi \right\} - U'' \phi + \frac{i\nu}{\alpha} \left\{ \phi^{IV} - 2(\alpha^2 + \beta^2) \phi'' + (\alpha^2 + \beta^2)^2 \phi \right\} = 0 \tag{25}$$

This is the standard form for the investigation of the stability of laminar motion. Numerous papers have been devoted to this subject; for a summary refer to: C.C. Lin, On the Stability of Two-Dimensional Parallel Flow, Quart. Appl. Math. 3 (1945-46), pp. 117-142, 218-234, 277-301. Usually β is taken zero, which does not change the essential character of the equation.

When viscosity is small, the terms multiplied by ν can be neglected in those parts of the field where there are no sharp gradients in ϕ . It is possible, however, that the factor $U - \omega/\alpha$, which multiplies the first term of the equation, vanishes for one or more values of z . If ω is real, the ratio ω/α gives the velocity of propagation of the disturbances in the direction of the x -axis, and a singularity appears where this velocity is equal to the velocity of the main flow. This circumstance has played a big part in all investigations, where the crucial point was to obtain solutions with real ω , separating the domain of damped solutions from that of increasing (unstable) solutions. Re-introduction of the terms with ν (at least of the most im-

portant term $i \phi^{IV}/a$) prevents the appearance of a logarithmic infinity in ϕ , but a concentrated layer of vorticity is obtained in the neighborhood of the critical value(s) of z .

Since the mathematical theory involved in these relations is very complicated, it does not look promising at the present moment to continue in this direction and to investigate the still more difficult question of the coupling between the various possible components. In the next chapter we shall turn to a simplified mathematical model, which will help us to obtain insight into certain aspects of the coupling problem.